Mathematical Reviews

Edited by

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Vol. 13, No. 5 May, 1952 pp. 419–520

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MATHEMATICAL REVIEWS

Published monthly, except August, by

THE AMERICAN MATHEMATICAL SOCIETY, Prince and Lemon Streets, Lancaster, Pennsylvania

Sponsored by

THE AMERICAN MATHEMATICAL SOCIETY
THE MATHEMATICAL ASSOCIATION OF AMERICA
THE INSTITUTE OF MATHEMATICAL STATISTICS
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Editorial Office

MATHEMATICAL REVIEWS, 80 Waterman St., Providence 6, R. I.

Subscriptions: Price $20 per year ($10 per year to members of sponsoring societies). Checks should be made payable to MATHEMATICAL REVIEWS. Subscriptions may be addressed to MATHEMATICAL REVIEWS, Lancaster, Pennsylvania, but should preferably be addressed to the American Mathematical Society, 80 Waterman St., Providence 6, R. I.

This publication was made possible in part by funds granted by the Carnegie Corporation of New York, the Rockefeller Foundation, and the American Philosophical Society held at Philadelphia for Promoting Useful Knowledge. Its preparation is also supported currently under a contract with the Office of Scientific Research, Headquarters, Air Research and Development Command, U. S. Air Force. These organizations are not, however, the authors, owners, publishers, or proprietors of this publication, and are not to be understood as approving by virtue of their grants any of the statements made or views expressed therein.

Entered as second-class matter February 3, 1940 at the post office at Lancaster, Pennsylvania, under the act of March 3, 1879. Accepted for mailing at special rate of postage provided for in paragraph (d-2), section 34.40, P. L. and R. of 1948, authorized November 9, 1940.

Mathematical Reviews

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HISTORY

✓ ***Selections illustrating the history of Greek mathematics. With an English translation by Ivor Thomas. Vol. I. From Thales to Euclid.** Harvard University Press, Cambridge, Mass.; William Heinemann, Ltd., London, 1951. xvi+505 pp.

✓ ***Selections illustrating the history of Greek mathematics. With an English translation by Ivor Thomas. Vol. II. From Aristarchus to Pappus.** Harvard University Press, Cambridge, Mass.; William Heinemann, Ltd., London, 1951. x+683 pp.

Reprinted with some revision from an earlier edition [vol. I, 1939; vol. II, 1941]. On facing pages are printed the original Greek text and the translation.

✓ ***Reidemeister, Kurt. Das exakte Denken der Griechen. Beiträge zur Deutung von Euklid, Plato, Aristoteles.** Claassen & Goverts, Hamburg, 1949. 108 pp. 7.50 DM.

Die vorliegende Schrift ist ein vorzüglicher Führer in die antike Geisteswelt, vornehmlich in die Untersuchungen zur Grundlegung der exakten Wissenschaften. Sie wird allen denjenigen willkommen sein, die sich veranlasst sehen, nach dem Ursprung unserer Denkformen zu suchen und dabei zu Euklid, Plato und Aristoteles geführt werden. Ausserdem enthält das Buch manchen wertvollen Beitrag zur Klärung spezieller mathematikgeschichtlicher Fragen. Nach einem einführenden Kapitel über "Mathematisches Denken" behandelt Verf. in Kapitel 2 "Die Arithmetik der Griechen". Es handelt sich vor allem darum, den pythagoräischen Ausspruch "alles ist Zahl" zu interpretieren. Zunächst werden die Quellen kritisch beleuchtet: Plato und die Neuplatoniker, Euklid. Sodann folgt eine Auseinandersetzung mit der Lehre von den Mitteln und den irrationalen Grössen. Dieser schliesst sich die Darstellung der "Lehre vom Geraden und Ungeraden" (Euklid IX, 21–36) an, in der Verf. mit O. Becker ein ältestes Stück griechischer Mathematik erblickt. Sie ist erstes Lehrstück einer logisch arithmetischen Wissenschaft und wird hierdurch das Fundament der pythagoräischen Metaphysik. Ihre Auswirkung auf Plato wird aufgezeigt. Sodann wird in der "Arithmetik der geometrischen Reihe", in der "Sectio canonis" und in "Euklid X" die Klassifizierung der Irrationalitäten behandelt: Alles dies sind streng arithmetische Theorien, womit Verf. sein Ziel erreicht. Das folgende, ebenso zentrale Kapitel, wendet sich der "Mathematik und Logik bei Plato" zu. Die "Wege zum Plato-Verständnis" geben dem weniger versierten Leser die Quellen und Kommentare, und es ist insbesondere für dieses Kapitel hervorzuheben, dass Verf. seine Ansichten ausführlich und sorgfältig belegt. Im Mittelpunkt der Auseinandersetzung steht die Interpretation des Sophistes und des Parmenides. Wenn dabei auch in Einzelheiten abweichende Beurteilung möglich ist, so wird doch jeder Leser der Grundthese des Verf. zustimmen: "Auf dem Denken Platos ruht der Glanz des Seins". Hierdurch wird das ontologische Problem der Mathematik in seiner ganzen Breite aufgerollt. Im folgenden Kapitel wird gezeigt, wie diese Problemstellung durch Aristoteles abgewandelt wird, die Mathematik wird zur Form, Stoffliches tritt in den Vordergrund des Interesses. Im Schlusskapitel "Geometrie und Kosmologie der Griechen kommt Verf. nochmals darauf zu sprechen, wie mathematisches und ontologisches Denken im Mittelpunkt griechischen Denkens stehen, indem dessen Funktion beim Aufbau der Kosmologie aufgezeigt wird. [Für einzelne kritische Bemerkungen zum Inhalt des Buches siehe das Referat von van der Waerden in Gnomon 22, 61–65 (1950)]. *J. J. Burckhardt* (Zürich).

Pihl, Mogens. The place of Theodoros in Plato's "Theaitetos" and the earliest history of irrational numbers. Mat. Tidsskr. A. 1951, 19–38 (1951). (Danish)

Summary of the modern attempts of reconstructing the method by which Theodoros demonstrated the irrationality of $\sqrt{2}, \cdots, \sqrt{17}$. *O. Neugebauer* (Providence, R. I.).

✓ ***Rome, A. The calculation of an eclipse of the sun according to Theon of Alexandria.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 1, pp. 209–219. Amer. Math. Soc., Providence, R. I., 1952.

The author gives a summary of some results reached in preparing his forthcoming edition of Theon's Commentary to Book VI of the Almagest. He uses as an example Theon's discussion of the solar eclipse of 364 June 16. Many remarks are made which are of general interest in the study of Greek astronomy. The following ones may be particularly mentioned. Delambre's scepticism concerning actual observations is unfounded (p. 211 and p. 218). Use of Egyptian fractions is generally a sign of rounded-off results as compared with the sexagesimally computed numbers; Ginzel's discussion of our eclipse in his Spezieller Kanon [Mayer und Müller, Berlin, 1899] is based on incorrect readings of Egyptian fractions (p. 211). Similarly, Fotheringham was mislead by the Basle edition of 1538 which changed seasonal hours into equinoctial hours. For the actual observation of solar eclipses the use of blue Alexandrian glass as a filter is considered as a possibility (p. 213). The arrangement of the tables of the Almagest in columns of 48 lines is taken over from Hipparchus, the only exception being the table of syzygies which is based on a 25-year cycle. In the "Handy Tables" this new arrangement became standard. And it is plausible that Ptolemy to a large extent used the help of other persons for the numerical computation of his tables (p. 215). The element called "prosneusis" in eclipses was used for the determination of the points of contact in actual observations (p. 218). *O. Neugebauer* (Princeton, N. J.).

✓ ***Bocheński, I. M. Ancient Formal Logic.** Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1951. ix+122 pp. 12 florins.

This little book, according to its author, "is intended to supply mathematical logicians with a synthetic outline of

the main aspects of ancient formal logic which are known in the present state of research." The author continues by explaining to us his conception of the meaning of some of the terms of the above quotation. Actually one gets the idea that the book is mainly a summary of the formal aspect of Aristotle's syllogistic, together with an indication of the influence on Aristotle's thinking by the researches of his predecessors, and of his influence on the ancient logical schools that follow him. The work is presented in the language and symbolism of present-day mathematical logic, and there are numerous references to source works, as well as to both ancient and modern commentaries.

E. J. Cogan (State College, Pa.).

Shankar Shukla, Kripa. On S'rīdhara's rational solution of $Nx^2+1=y^2$. Gaṇita 1, 1–12 (1950).

The author considers a method, with extensions, of the Hindu mathematician Śrīdhara for rational solutions of $Nx^2\pm1=y^2$, $1-Nx^2=y^2$, $Nx^2\pm C=y^2$, and $C-Nx^2=y^2$. The method is given in the Pāṭīgaṇita, as yet unpublished and existing only in two incomplete manuscripts. This rule of Śrīdhara is stated to be different from that of other Hindu mathematicians who have considered some of the same equations and includes the later results of the Europeans John Wallis and W. Brouncker.

E. B. Allen.

Venkatachalam Iyer, R. Pāṭīgaṇita and the Hindu abacus. Math. Student 18, 79–82 (1950).

Sinha, Sri Rama. Bhaskara's Lilavati. Bull. Allahabad Univ. Math. Assoc. 15, 9–16 (1951).

Prasad, B. N., and Shukla, R. Aryabhata of Kusumpura. Bull. Allahabad Univ. Math. Assoc. 15, 24–32 (1951).

Dixit, K. R. The history of Indian astronomy. Scientia 45, 315–318 (1951).

Agostini, Amedeo. Un codice di aritmetica anonimo del sec. XV. Boll. Un. Mat. Ital. (3) 6, 231–240 (1951).

Shirley, John W. Binary numeration before Leibniz. Amer. J. Phys. 19, 452–454 (1951).

Abellanas, Pedro. Historical essay on the concepts of space and geometry. Revista Acad. Ci. Zaragoza (2) 6, 9–26 (1951). (Spanish)

Procissi, Angiolo. Il caso irriducibile dell'equazione cubica da Cardano ai moderni algebristi. Period. Mat. (4) 29, 263–280 (1951).

Pati, Tribikram. The development of non-Euclidean geometry during the last 150 years. Bull. Allahabad Univ. Math. Assoc. 15, 1–8 (1951).

Ricci, Giovanni. La scuola matematica pisana dal 1848 al 1948. Rivista Mat. Univ. Parma 2, 155–174 (1951).

González, Mario O. Evolution of mathematics in the modern era. Revista Soc. Cubana Ci. Fís. Mat. 2, 150–158 (1950). (Spanish)

Weyl, Hermann. A half-century of mathematics. Amer. Math. Monthly 58, 523–553 (1951).

The aim of this paper is to point out general trends of modern mathematics, to explain outstanding notions and to list important problems solved during the last fifty years. The importance of the axiomatic approach is stressed. The topics treated are as follows. Rings, fields, ideals. Primadic numbers. Distribution of primes and the zeta function. Waring's theorem. Transcendental numbers. Groups, vector spaces, algebras. Jordan-Hölder theorem. Group representations. Lie groups. Integral equations, linear operators in Hilbert space, quantum mechanics. Almost periodic functions. Lebesgue's integral, measure and probability, ergodic theory. Calculus of variations, Dirichlet's principle, uniformization. Topology: homology and cohomology, Alexander's duality theorem, simplicial approximation, dimension, mapping degree, fixed points, universal covering space, fibre spaces. Harmonic integrals. Banach spaces. Minimal surfaces. Calculus of variations in the large. Meromorphic functions. Axiomatic geometry without points. Riemannian geometry and relativity. Differential geometry in the large. Foundations of mathematics.

B. L. van der Waerden (Zurich).

de Broglie, Louis. Un mathématicien, homme de lettres: d'Alembert. Rev. Hist. Sci. Appl. 4, 204–212 (1951).

Jessen, Børge. Harald Bohr, 22 April 1887–22 January 1951. Mat. Tidsskr. A. 1951, 1–18 (1951). (Danish)
Jessen, Børge. Harald Bohr, 22 April 1887–22 January 1951. Acta Math. 86, I–XXIII (1951).

A memorial address discussing Bohr's life and work. The English version in Acta Math. is a translation of the Danish version except for minor changes. However, it is followed by a list of Bohr's published work.

Bochner, Salomon. Obituary: Harald Bohr, April 22, 1887–January 22, 1951. Bull. Amer. Math. Soc. 58, 72–75 (1952).

Hofmann, J. E. Zum Gedenken an Thomas Bradwardine. Centaurus 1, 293–308 (1951).

***Polnoe sobranie sočinenii P. L. Čebyševa. Tom V. Pročie sočineniya. Biografičeskie materialy. [Complete Collected Work of P. L. Čebyšev. Vol. V. Other Works. Biographical Materials].** Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1951. (3 plates) 474 pp.

For a note concerning vols. 1–4 see these Rev. 11, 150.

Agostini, Amedeo. La convergenza delle serie e una memoria di Giuliano Frullani. Period. Mat. (4) 29, 241–248 (1951).

Bouligand, Georges. En hommage à F. Gonseth. À l'occasion de son soixantième anniversaire. Revue Sci. 89, 243–244 (1951).

Pécaut, Catherine. L'oeuvre géologique de Leibniz. Rev. Gén. Sci. Pures Appl. 58, 282–296 (1951).

Taton, René. Monge, créateur des coordonnées axiales de la droite, dites de Plücker. Elemente der Math. 7, 1–5 (1952).

Gnedenko, B. V. Mihail Vasil'evič Ostrogradskiĭ. Uspehi Matem. Nauk (N.S.) 6, no. 5(45), 3–25 (1 plate) (1951). (Russian)

Rabinovič, Yu. L. The integral theorem of M. V. Ostrogradskiĭ. Uspehi Matem. Nauk (N.S.) 6, no. 5(45), 26–32 (1951). (Russian)

★Œuvres de Henri Poincaré. Publiées sous les auspices de l'Académie des Sciences par la Section de Géométrie. Tome IV. Publié avec la collaboration de Georges Valiron. Gauthier-Villars, Paris, 1950. iii+632 pp.

This volume of Poincaré's collected work, the second part on pure analysis, brings together his papers on the theory of functions of one and two complex variables, abelian functions and hyperfuchsian functions, as well as several papers on trigonometric series having their origin in astronomical problems. G. Valiron has added notes on the papers, tracing the further development of many of the ideas. In addition, the pertinent parts of Poincaré's own analysis of his papers is included [Acta Math. **38**, 3–135 (1921)]. Volume 1 was issued in 1928, vol. 2 in 1916, vol. 3 in 1934; seven volumes in all are anticipated.

★Œuvres de Henri Poincaré. Publiées sous les auspices de l'Académie des Sciences par la Section de Géométrie. Tome V. Publié avec la collaboration de Albert Châtelet. Gauthier-Villars, Paris, 1950. viii+552 pp.

This volume contains Poincaré's papers on algebra and number theory as well as his own analysis of these papers [Acta Math. **38**, 3–135 (1921), especially pp. 89–90, 92–100]. A. Châtelet has added notes at the ends of the various sections.

★Châtelet, Albert, Valiron, Georges, LeRoy, Edouard, et Borel, Émile. Hommage à Henri Poincaré. Congrès International de Philosophie des Sciences, Paris, 1949, vol. I, Épistémologie, pp. 37–64. Actualités Sci. Ind., no. 1126. Hermann & Cie., Paris, 1951.

Vučkić, Milenko. Poncelet et la théorie de la meilleure approximation. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **6**, 115–121 (1951). (Serbo-Croatian. French summary)

Kuroš, A. G. Otto Yul'evič Šmidt (for his sixtieth birthday). Uspehi Matem. Nauk (N.S.) **6**, no. 5(45), 197–199 (1 plate) (1951). (Russian)

v. Laue, M. Sommerfelds Lebenswerk. Naturwissenschaften **38**, 513–518 (1951).

Jones, P. S. Brook Taylor and the mathematical theory of linear perspective. Amer. Math. Monthly **58**, 597–606 (1951).

Mardžanišvili, K. K. Ivan Matveevič Vinogradov (for his sixtieth birthday). Uspehi Matem. Nauk (N.S.) **6**, no. 5(45), 190–196 (1 plate) (1951). (Russian)

de Finetti, Bruno. L'opera di Abraham Wald e l'assestamento concettuale della statistica matematica moderna. Statistica, Milano **11**, 185–192 (1951).

Morgenstern, Oskar. Abraham Wald, 1902–1950. Econometrica **19**, 361–367 (1951).

Roy, S. N. Obituary: Abraham Wald. Calcutta Statist. Assoc. Bull. **3**, 133–138 (1951).

FOUNDATIONS

★Péter, Rózsa. Rekursive Funktionen. Akademischer Verlag, Budapest, 1951. 206 pp.

This is the first book to appear on the theory of recursive functions, a topic of central interest to the foundations of mathematics and one which aroused general interest in its application to the incompleteness of arithmetic systems in the work of Gödel. While the author devotes one section to the history and applications of the theory and the final three sections to topics primarily of foundational interest, her major concern is the arithmetic aspect of the subject, especially the classification of types of recursive functions and theorems on the reduction of the schemata required to generate various classes of functions. In this field the book will be a valuable and authoritative reference work. The study proceeds through the consideration of primitive recursive functions to general recursive functions, discussing in detail the scale of k-fold recursive functions. A large part of the theory is made up of the author's own work which has appeared in a series of papers since 1932.

The sections of the book are as follows: (1) Familiar examples of functions defined by recursion from various portions of mathematics. (2) The concepts of primitive recursive function and predicate and the familiar results on bounded quantification. (3–6) Discussion of various types of recursive definition such as course-of-values and simultaneous recursion and recursion in several variables. These and other types are shown to be reducible to primitive recursion. (7) Reduction of the schemata required to define the class of primitive recursive functions. An account is given of the work of Robinson [Bull. Amer. Math. Soc. **53**, 925–942 (1947); these Rev. **9**, 221] showing that recursions can be limited to iterations of one place functions on 0 using three additional functions. (8) The author characterizes as arithmetic functions, those which can be built up from addition, multiplication, and natural-number subtraction and division. It is shown that the iteration of the power function gives a primitive recursive function with values which increase more rapidly than those for any elementary function. (9) This approach is extended to give the Ackermann function which is recursive but not primitive recursive. (10–14) Discussion of k-fold (eingeschachtelte) recursion and the relation of these to functions defined by transfinite recursion. A diagonal argument shows that $(k+1)$-fold recursions are not reducible to k-fold recursions. This work appeared in the author's recent paper [J. Symbolic Logic **15**, 248–272 (1950); these Rev. **12**, 469]. In these sections the author also discusses recursions of higher order involving functionals with functions rather than numbers as values. Finally a normal form for k-fold recursion is established. (15) Gödel numbering of defining equations and deductions from them. (16) The concept of general recursive function. (17) An outline of the theory of Kleene [Math. Ann. **112**, 727–742 (1936)] resulting in a normal form for general recursive functions. Kleene's result that all general recursive functions can be built up from $+$, $\cdot$, and the Kronecker δ-function, using the least number operator and substitution is stated. In this connection the reader should note the work of Julia Robinson [Proc. Amer. Math. Soc. **1**, 703–718 (1950); these Rev. **12**, 469] which has appeared since the publication of Peter's book. (18) A discussion of alternatives to Kleene's normal form for general recursive functions, the results of Skolem and Markov on this matter. (19) Kleene's

non-recursive function. (20) The notion of a mechanically computable function and the Turing machines. It is shown that any function computable by a Turing machine is general recursive. (21) An excellent brief history and discussion of applications to logic. (22–23) Effectively undecideable problems. It is shown that there exists no decision procedure for establishing which systems of equations define general recursive functions; and, using the elegant proof of Skolem, that there is no general decision procedure for arithmetic problems. (24) The work of Goodstein and Specker in application of recursive functions to analysis is sketched.

The book is clearly and concisely written and will be useful as an introduction to the subject as well as a reference work. An extensive bibliography is appended.

D. Nelson (Washington, D. C.).

✓★**Kleene, S. C. Recursive functions and intuitionistic mathematics.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 1, pp. 679–685. Amer. Math. Soc., Providence, R. I., 1952.

The author sketches a method by which it is hoped that his interpretation of intuitionistic number theory by means of a "realization" [J. Symbolic Logic **10**, 109–124 (1945); these Rev. **7**, 406] may be extended to intuitionistic set theory, in particular, to Brouwer's main theorem on bounded spreads (finite Mengen) [Math. Ann. **97**, 60–75 (1927); p. 66]. Two new ideas are involved: (i) that of a function $\varphi(\alpha_i, x_k)$ uniformly recursive in other functions α_i; (ii) the extension of the notion of realizability to formulas involving function variables α_i with corresponding quantifiers and symbols for functions $\varphi(\alpha_i, x_k)$. For example, $(E\beta)A(\alpha, \beta)$ should be realizable if and only if there is a function β, general recursive in α, for which $A(\alpha, \beta)$ is realizable. The function variables α_i are intended to represent Brouwer's choice sequences, that is, arbitrary number-theoretic functions. It is shown by an example that Brouwer's theorem fails if these variables are restricted to range over general recursive functions only. It follows that for the intended interpretation the function variables with corresponding quantifiers are indispensable. *A. Heyting* (Amsterdam).

Quine, W. V. On the consistency of "New Foundations." Proc. Nat. Acad. Sci. U. S. A. **37**, 538–540 (1951).

The paper is concerned with a relationship between the author's system "New foundations for mathematical logic" [Amer. Math. Monthly **44**, 70–80 (1937)] and that form of the Principia Mathematica (without axiom of infinity or axiom of choice) in which relational predication is defined in terms of set membership and variables are ambiguous as to type. Doubts as to the consistency of the former system have been expressed by various authors. The author gives here an outline of an argument purporting to show that the stratified theorems of his system are the same as those of the stated form of the Principia Mathematica, and that therefore the two systems are equivalent so far as consistency is concerned. This is of interest because other forms of the Principia are known to be consistent.

H. B. Curry (State College, Pa.).

Lyndon, R. C. Identities in two-valued calculi. Trans. Amer. Math. Soc. **71**, 457–465 (1951).

All two-valued logics have been enumerated by Post [The two-valued iterative systems of mathematical logic, Princeton Univ. Press, 1941; these Rev. **2**, 337]. Lyndon obtains a finite set of axioms for each of these treated as an algebraic system. As it is unnecessary to distinguish between equivalent algebras he considers only one algebra corresponding to each of Post's O_4, O_9, S_1, S_4, S_3, S_6, A_4, A_2, A_1, L_2, L_1, C_2, C_1, L_4, L_5, F_4, $F^n{}_4$ $(n \geqq 3)$, F_8, F_7, F_5, C_4, $F^n{}_8$, $F^n{}_7$, $F^n{}_5$, D_2, D_3, D_1.

It is first shown that every system containing a function corresponding to implication is axiomatisable. After remarking that a complete set of axioms for each of $O_4, \cdots, L_5$ can be obtained from the axioms for Boolean algebras and Boolean rings, and that as systems F_4, $F^n{}_4$ contain a function corresponding to implication they are axiomatisable, Lyndon shows that the systems $F_8, \cdots, F^n{}_5$ are axiomatisable. In order to do this he proves a representation theorem patterned after that of Stone [Trans. Amer. Math. Soc. **40**, 37–111 (1936)]. Finally, the axiomatisability of D_1, D_2, D_3 is deduced from that of $F^3{}_7$. *A. Rose.*

✓★**Curry, Haskell B. Outlines of a Formalist Philosophy of Mathematics.** Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1951. viii+75 pp. 7.50 florins.

This is an excellent book whose contents are described very accurately by its title. It is based on a paper submitted to the International Congress for the Unity of Science held in Cambridge, Mass., 1939. As is pointed out in the preface, it thus reflects the opinions the author held over 10 years ago, which may have changed somewhat in the meantime. It is written from the point of view of a mathematician, but in an attempt to meet the philosophers half way.

"The central problem in the philosophy of mathematics is the definition of mathematical truth. If mathematics is to be a science, then it must consist of propositions concerning a subject matter, which propositions are true insofar as they correspond with the facts. We are concerned with the nature of this subject matter and these facts. The ordinary mathematician bases the idea of truth upon that of rigor; he regards a mathematical proposition as true when he has a rigorous proof of it." It is the thesis of this book that an objective criterion of truth and rigor can be found. There are three points of view on the nature of the subject matter of mathematics: (1) realism, which takes as subject matter the nature of the world around us; (2) idealism, which deals with mental objects of some sort (Platonism and Intuitionism); (3) formalism. Realism is not now taken very seriously by mathematicians. Idealism is shown to lead to rather vague truth criteria which are largely dependent on metaphysics, so we now turn to the formalist point of view. In the following, the words "proposition", "true", "predicate" are used in the sense of ordinary discourse. "Theorem" will be used in the sense of "true proposition". A formal system is then "defined by a set of conventions, which I shall call its primitive frame, specifying the following: first, a set of objects, which I shall call terms; second a set of propositions, which I shall call elementary propositions, concerning these terms; and third, which of these elementary propositions are true. The first and third of these specifications are essentially recursive definitions." The terms comprise tokens or primitive terms, operations (i.e. modes of combination for forming new terms) and rules of formation specifying how these operations are to be applied to already existing terms in order to form new terms. The elementary propositions are formed by applying certain specified elementary predicates to sequences of terms of the proper number and kind. Arithmetic, formal syntax, two-valued propositional algebra, Church's calculus of λ-conversion, Gödel's system P, combinatory logic and elementary theory of polynomials

are specific examples of formal systems. The tokens may have various characters: they may for instance be specific objects such as buttons, or symbols of an underlying object language, or variables. It is possible to simplify the definition of a formal system by the following reductions: first, by adding new operations it is possible to reduce the number of predicates to one unary predicate, then, by reductions due to Schönfinkel, we can reduce the operations to one binary one and the number of tokens to one. Curry shows that the notion of a formal system is essentially equivalent to that of a syntax of a language. He criticises the extreme syntactic point of view, mostly because it leads to needless complications many of which are merely linguistic problems. We study formal systems by any means at our command. "In so doing, we may consider propositions of a more complex nature; let us call these metapropositions and the method of study which gives rise to them the metatheoretic method." Metapropositions are therefore stated in the same language as elementary propositions. However, as pointed out by Kleene (in his review [J. Symbolic Logic **6**, 100–102 (1941)] of "Some aspects of the problem of mathematical rigor," by H. B. Curry [Bull. Amer. Math. Soc. **47**, 221–241 (1941); these Rev. **2**, 340]), the impression that all distinction between object language and syntax language can be eliminated is somewhat misleading. Curry defines mathematics as the science of formal systems. This view does not involve any metaphysical bias "and is therefore compatible with practically any sort of philosophy. It is the only conception so far proposed which has that character." We now turn to the relation of mathematics to its applications. It is here that philosophical differences emerge, for different philosophers will consider different types of formal systems as acceptable. "Acceptability is a matter of interpretation of the formal system in relation to some subject matter. . . . Thus an intuitionist would say that those and only those formal systems are acceptable for which an interpretation exists such that the premises have the proper sort of intuitive evidence; a Platonist would make a corresponding statement from his point of view and so on." We now come to the question of logic in general: ". . . Let us distinguish two senses of 'logic'. On the one hand logic is that branch of philosophy in which we discuss the nature and criteria of reasoning; in this sense let us call it logic (1). On the other hand in the study of logic (1) we may construct formal systems having an application therein; such systems and some others we often call 'logics'. . . . There is no one system of logic which is acceptable a priori for every purpose under the sun. . . . The acceptability of systems of logic is essentially an empirical matter. This revolution in our logical conceptions is to be attributed to the rise of the formalist mathematical point of view." *I. L. Novak.*

***Tarski, Alfred. A decision method for elementary algebra and geometry.** 2nd ed. University of California Press, Berkeley and Los Angeles, Calif., 1951. iii+63 pp. $2.75.

A photographic reprint of the first edition [The RAND Corporation, Santa Monica, Calif., 1948; these Rev. **10**, 499]. Known errors have been corrected and three pages of supplementary notes have been added.

Stenius, Erik. Das Problem der logischen Antinomien. Soc. Sci. Fenn. Comment. Phys.-Math. **14**, no. 11, 89 pp. (1949).

The author believes that it is possible to resolve the logical antinomies (Russell, Grelling, etc.) by a critical examination of the way in which they arise, without recourse to a formal language. An antinomy is based on the assumption that the concepts involved are free from contradiction. Considering the various antinomies in turn, the author shows that this assumption is unwarranted since circle definitions of one form or another are used. The theory of types is rejected as arbitrary. The formulations of set theory are developed from a genetic-constructive point of view, the domain of sets being regarded as an expanding organism whose growth is ensured by a hierarchy of circle-free (but possibly many-valued) definitions. The author states that the problem of the excluded middle can also be resolved by his method. There is much that is interesting in the author's general approach and in his arguments but a formal logician may feel that a number of these are inconclusive.

A. Robinson (Toronto, Ont.).

***Beth, E. W. L'état actuel du problème logique des antinomies.** Congrès International de Philosophie des Sciences, Paris, 1949, vol. II, Logique, pp. 7–14. Actualités Sci. Ind., no. 1134. Hermann & Cie., Paris, 1951.

An expository article, essentially an abbreviation of Livre V of the author's book "Les fondements logiques des mathématiques" [Gauthier-Villars, Paris, 1950; these Rev. **12**, 71]. *H. B. Curry* (State College, Pa.).

***Dieudonné, Jean. L'axiomatique dans les mathématiques modernes.** Congrès International de Philosophie des Sciences, Paris, 1949, vol. III, Philosophie Mathématique, Mécanique, pp. 47–53. Actualités Sci. Ind., no. 1137. Hermann & Cie., Paris, 1951.

***Bouligand, Georges. Connaissance mathématique, idées de construction et d'existence.** Congrès International de Philosophie des Sciences, Paris, 1949, vol. III, Philosophie Mathématique, Mécanique, pp. 75–83. Actualités Sci. Ind., no. 1137. Hermann & Cie., Paris, 1951.

***Apéry, Roger. Le rôle de l'intuition en mathématiques.** Congrès International de Philosophie des Sciences, Paris, 1949, vol. III, Philosophie Mathématique, Mécanique, pp. 85–88. Actualités Sci. Ind., no. 1137. Hermann & Cie., Paris, 1951.

***Borel, Émile. La définition en mathématiques.** Congrès International de Philosophie des Sciences, Paris, 1949, vol. III, Philosophie Mathématique, Mécanique, pp. 89–99. Actualités Sci. Ind., no. 1137. Hermann & Cie., Paris, 1951.

***Denjoy, Arnaud. Récurrence et antirécurrence.** Congrès International de Philosophie des Sciences, Paris, 1949, vol. III, Philosophie Mathématique, Mécanique, pp. 101–106. Actualités Sci. Ind., no. 1137. Hermann & Cie., Paris, 1951.

Nicolescu, Miron. Sur la notion de nombre naturel. Acad. Repub. Pop. Române. Bul. Şti. A. **1**, 205–212 (1949). (Romanian and French)

The author objects to the ordinary formulation of natural arithmetic in terms of Peano postulates and recursive definitions of sum and product on the ground that the commutative law of addition, which from the intuitive point of view is the most natural property of numbers, is so hard to derive. Instead he proposes to base the theory of natural numbers on the theory of cardinal numbers, combined with an inductive

definition of the natural numbers much as in the Principia. There are no new results. *H. B. Curry.*

★**Borel, Émile. Les nombres inaccessibles. Avec une note de Daniel Dugué.** Gauthier-Villars, Paris, 1952. x+141 pp. 1200 francs.

A collection of observations on: definability and possible cognizance of individual numbers and sets of numbers; bases of numeration; probability; the history of the theory of measure; cardinal numbers; and assorted other topics of a philosophico-mathematical nature. *E. Hewitt.*

★**Lupasco, Stéphane. L'axiome du choix, le principe de Pauli et le phénomène vital.** Congrès International de Philosophie des Sciences, Paris, 1949, vol. I, Épistémologie, pp. 153–161. Actualités Sci. Ind., no. 1126. Hermann & Cie., Paris, 1951.

★**Bernays, Paul. Logique et science.** Congrès International de Philosophie des Sciences, Paris, 1949, vol. II, Logique, pp. 1–5. Actualités Sci. Ind., no. 1134. Hermann & Cie., Paris, 1951.

Author views with caution such trends as the elimination of negation and the specialization of logic for specific sciences. *C. C. Torrance* (Monterey, Calif.).

★**Wilder, R. L. The cultural basis of mathematics.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 1, pp. 258–271. Amer. Math. Soc., Providence, R. I., 1952.

The author is concerned with two main questions: (1) How does culture (in its broadest sense) determine a mathematical structure, such as a logic? (2) How does culture influence the successive stages of the discovery of a mathematical structure? These questions are answered by examples, such as Intuitionism for (1), and symbolism for (2). The use of mathematics as a clew to, and as a study-method of, culture analysis is indicated. *C. C. Torrance.*

Greenwood, Thomas. Les mathématiques qualitatives. Rev. Trimest. Canad. **37**, 287–308 (1951).

Chisini, O. Il valore sociale della matematica. Period. Mat. (4) **29**, 255–262 (1951).

★**Destouches-Février, Paulette. La structure des théories physiques.** Presses Universitaires de France, Paris, 1951. xi+423 pp.

This book is a systematic and comprehensive treatment of the theory developed recently in numerous papers by J.-L. Destouches, the author, and others. In this treatment the primitive concept is that of experimental proposition (proposition expressing the result of a measurement of a physical magnitude). It is shown how the logic (laws of combination) of experimental propositions is determined by the physical domain being investigated. Because of the relation existing between physical properties and experimental propositions, it follows that each physical theory must employ a logic suited to it. Since a physical theory may involve other novel features besides its logic, the question arises as to what are the general features of a physical theory. The author considers that a physical theory has two essential features: its form must be deductive, and its content must be verifiable. The first of these conditions is treated by the theory of axiomatics, and the second by the theory of prediction (which is developed in some detail). Conditions are given that a physical theory be deterministic or indeterministic. As an application of these general considerations, certain aspects of the present theory of wave mechanics are shown to be inescapable. *C. C. Torrance.*

ALGEBRA

★**Sperner, Emanuel. Einführung in die analytische Geometrie und Algebra. 2 Teil.** Vandenhoeck & Ruprecht, Göttingen, 1951. 389 pp. 37.40 DM; bound 43.00 DM.

This is the second edition of volume II of an earlier book by Schreier and Sperner with the same title [Teubner, Leipzig, 1935], the second edition of volume I having appeared in 1948. It is divided into four sections: (I) Algebraic fields (49 pages); (II) Elements of group theory (31 pages), (III) Linear transformations, matrices (67 pages); and (IV) Projective geometry (234 pages). In comparison with the first edition parts I and II are much the same while part III has been shortened and part IV has been expanded.

The text is readable and clear throughout. Although the subject matter is moderately advanced, there has been a conscious effort to avoid any greater generality than what is needed for the topics considered. Thus (page 32) ideals are defined for polynomial rings, but not for rings in general. Again (page 68) a factor group is defined, but no mention is made of homomorphism. On the other hand, points often glossed over are well brought out. It is shown (pp. 10–11) that the associative law for addition implies that the sum of n elements in a given order is the same for all bracketings. The examples in many places are valuable supplements to the text. Quaternions appear in an example on page 111, and in two examples on pages 149–150 the reduction of symmetric matrices to real diagonal form is based solely on the fundamental theorem of algebra, rather than on the Jacobi procedure given in the text which involves considerable detailed study of limits of sequences of matrices.

The longest section of the book is that devoted to projective geometry. The approach is entirely that of analytic geometry and there are no synthetic proofs. There is an appropriate stress placed upon cross-ratio which is defined even for the mixed combination of two points and two hyperplanes. This is n-dimensional throughout and includes an excellent treatment of quadratic hypersurfaces.

Marshall Hall (Washington, D. C.).

★**Stojaković, M. La généralisation du théorème de Laplace et son application à la détermination du maximum du module du déterminant.** Premier Congrès des Mathématiciens et Physiciens de la R.P.F.Y., 1949. Vol. II, Communications et Exposés Scientifiques, pp. 149–155. Naučna Knjiga, Belgrade, 1951. (Serbo-Croatian. French summary)

An expository article. *A. W. Goodman.*

Silva, Joseph A. A theorem on cyclic matrices. Duke Math. J. **18**, 821–825 (1951).

Perfectionnement d'un théorème connu: (1) Si A est une matrice cyclique d'ordre n: $A=\|a_{j-i+1}\|$; $i, j=1, 2, \cdots, n$; $a_r=a_s$ pour $r\equiv s \pmod n$, le déterminant de A a pour développement: $d(A)=\prod_{i=1}^{n}\sum_{j=1}^{n}\alpha_i^{j-1}a_j$, où les α_i parcourent les racines $n$ièmes de l'unité. Si l'ordre n d'une telle

matrice A est divisible par un nombre premier p, on peut décomposer A en p^2 sous-matrices carrées A'_{j-i+1}. Alors A est encore cyclique par rapport aux A'. Multipliant A par deux matrices dont le déterminant est égal à l'unité et dont les éléments non nuls sont ceux du triangle de Pascal, puis développant le produit, l'auteur parvient à deux théorèmes dont le second généralise (1) de la manière suivante: Si $n=p^t\cdot m$, m premier avec p, alors $d(A)\equiv\prod_{i=1}^m(\sum_{j=1}^m\alpha_1^{j-1}a_j)^{p^t}$ (mod p), où les α_i sont les racines $m$ièmes de l'unité dans un champ de Galois convenablement choisi. La démonstration repose sur les deux congruences: $\binom{a}{c}\equiv\binom{a+hp}{c}$ (mod p); p premier; $0\leqq c<p$ et $\binom{a+hp}{p}\equiv h$ (mod p). Si $t=m=1$, on retrouve un résultat de O. Ore [même J. **18**, 343–354 (1951); ces Rev. **13**, 98]. *A. Sade* (Marseille).

Perfect, Hazel. On matrices with positive elements. Quart. J. Math., Oxford Ser. (2) **2**, 286–290 (1951).

Let A be a matrix all of whose elements are real and positive and set $\mu_i^{(r)}=(e_i'A^{r+1}u)/(e_i'A^ru)$ for u a column vector with positive elements and e_i the ith basis vector. Then, as $r\to\infty$, max $\mu_i^{(r)}$ decreases and min $\mu_i^{(r)}$ increases toward a common limit, the maximum root χ of A. Requiring A symmetric too, the projection of the unit vectors $(A^ru)/|A^ru|$ onto the (positive) characteristic vector of χ approaches unit length. *W. Givens* (Knoxville, Tenn.).

Shoda, Kenjiro. Über den Kommutator der Matrizen. J. Math. Soc. Japan **3**, 78–81 (1951).

Let A be a matrix of determinant one, with elements in a field K and with N the maximum degree of any of the irreducible factors of its characteristic determinant. Then A can be written as a product of at most N commutators of matrices with elements in K. The proof depends on a reduction to rational canonical form leading to a factorization $A=CB$ with C of determinant one, and with elements (involving parameters) and eigenvalues in K. As a lemma it is shown that if the characteristic determinant of A is separable, galois and has a root of the form β/β', where β' is a conjugate of β, then A is a commutator. In the second equation on page 80 a factor $B^{(0)}$ is omitted at the end. *W. Givens* (Knoxville, Tenn.).

Flanders, Harley. Elementary divisors of AB and BA. Proc. Amer. Math. Soc. **2**, 871–874 (1951).

Let L and M be vector spaces over a field k, A an endomorphism of L, B an endomorphism of M. If α is an element of k which is a root of the characteristic equation of A, the geometric multiplicity of α is the dimension of the subspace of L annihilated by $\alpha I-A$, where I is the identity endomorphism. The geometric multiplicities of the nonzero characteristic roots of AB coincide with those of BA. Furthermore, the elementary divisors of AB which do not have zero as a root coincide with the elementary divisors of BA. The proof of this is based on Fitting's lemma. Suppose now that 0 is a characteristic root of AB and that $t^{n_1}, t^{n_2}, \cdots$ are the elementary divisors corresponding to this root. This may be indicated by the sequence $n_1\geqq n_2\geqq\cdots$. Similarly, let the corresponding sequence for BA be denoted by $n_1'\geqq n_2'\geqq\cdots$. The author shows that $|n_j-n_j'|\leqq 1$. If C and D are endomorphisms of L and M respectively, the above conditions for C and D [in place of AB and BA] also are sufficient to assure the existence of a linear transformation A of L into M and a linear transformation B of M into L such that $AB=C$ and $BA=D$. Another related result is also obtained. *N. H. McCoy* (Northampton, Mass.).

Abstract Algebra

★Dubreil, Paul. Les méthodes modernes en algèbre. Congrès International de Philosophie des Sciences, Paris, 1949, vol. III, Philosophie Mathématique, Mécanique, pp. 55–65. Actualités Sci. Ind., no. 1137. Hermann & Cie., Paris, 1951.

Carruth, Philip W. Sums and products of ordered systems. Proc. Amer. Math. Soc. **2**, 896–900 (1951).

In this paper the methods of the reviewer [Trans. Amer. Math. Soc. **58**, 1–43 (1945); these Rev. **7**, 1] are applied to yield conditions on the index system and on the terms of an ordered sum or the factors of an ordered product of partially ordered systems necessary and sufficient that that sum or product be a system of a given type, for example a lattice, complete lattice, or ordinal number. *M. M. Day.*

Châtelet, Albert. Une forme générale des théorèmes de Schreier et de Jordan-Hölder. C. R. Acad. Sci. Paris **233**, 1165–1166 (1951).

The paper contains a simplification of certain earlier results [Ann. Sci. École Norm. Sup. (3) **64**, 339–368 (1948); these Rev. **10**, 181] by the same author on the generalisation of the theorem of Jordan-Hölder to systems of relations. *O. Ore* (New Haven, Conn.).

Kimura, Naoki. On latticoids. J. Sci. Gakugei Fac. Tokushima Univ. **1**, 11–16 (1950).

A "latticoid" is defined as a system L with two binary, commutative, associative operations which satisfy $x\cup x=x\cap x$ and $x\cup(x\cap y)=x\cap(x\cup y)=\sigma(x)$ for all $x, y\epsilon L$. In any latticoid L, the set L_σ of all $\sigma(x)$ is a lattice, which contains every sublatticoid-lattice of L, and the correspondence $x\to\sigma(x)$ is homomorphic. The definitions $a\vee b=\sigma(a\cup b)$ and $a\wedge b=\sigma(a\cap b)$ associate with any latticoid L a unique "simple" latticoid M, in which $a\vee b\epsilon M_\sigma$ and $a\wedge b\epsilon M_\sigma$ for all $a, b\epsilon M$. *G. Birkhoff* (Cambridge, Mass.).

Iseki, Kiyoshi. On closure operation in lattice theory. Nederl. Akad. Wetensch. Proc. Ser. A. **54** = Indagationes Math. **13**, 318–320 (1951).

Postulational study of closure operations [cf. A. Monteiro, Portugaliae Math. **4**, 158–160 (1945); these Rev. **7**, 8]. It is shown, for example, that an operation $A\to\bar{A}$ in a join-semilattice is a closure operation, if and only if it satisfies the identity $\bar{A}\cup B\leqq\bar{A}\cup\bar{B}$. *G. Birkhoff.*

Maeda, Fumitomo. Representations of orthocomplemented modular lattices. J. Sci. Hiroshima Univ. Ser. A. **14**, 93–96 (1950).

Let L be the lattice of linear subspaces of a projective geometry of finite dimension over a skewfield G. Then every involutory dual isomorphism $a\to a^\perp$ in L corresponds to a definition of orthogonality of vectors with components in G and this orthogonality can always be expressed through a suitable involutory anti-isomorphism $\alpha\to\alpha^*$ of G together with a positive definite Hermitian form: $(a_i)\perp(b_i)$ if and only if $\sum_i a_i^*\varphi_i b_i=0$. This was shown by G. Birkhoff and J. von Neumann [Ann. of Math. (2) **37**, 823–843 (1936), pp. 837–843]. The same theorem is now proved for the more general case that L is an orthocomplemented modular lattice of finite order $n\geqq 4$ over its von Neumann auxiliary ring G. The Birkhoff-von Neumann proof deals with L as a set of linear subspaces only; the present author simplifies the proof by using von Neumann's identification of the

elements of L with the right-ideals of a matrix ring along with the theorems of von Neumann concerning dual and anti-isomorphisms. The author needs the restriction on L that it have a homogeneous basis of mutually orthogonal elements but points out that this holds if L is a continuous geometry (the reviewer notes that more generally, it is sufficient that perspectivity be transitive in L).

I. Halperin (Kingston, Ont.).

Utz, W. R. Powers of a matrix over a lattice. Proc. Amer. Math. Soc. **2**, 305–306 (1951).

The author remarks that, if $M=\{a_{ij}\}$ is a square matrix with elements in a lattice and $a_{ij}\subset a_{mp}$ for $i\geqq m$ and $j\geqq p$, then $M\subset M^2=M^3=\cdots$. *L. Nachbin* (Rio de Janeiro).

Haimo, Franklin. A representation for Boolean algebras. Amer. J. Math. **73**, 725–740 (1951).

Two new concepts are discussed: that of vector-ordered group and that of Boolean group. The former includes, typically, any direct sum of Abelian groups in which $a>b$ is defined to mean that b can be obtained from a by replacing some non-zero components of a by zeros. In particular, it is shown that every Boolean algebra A is isomorphic with the lattice of all projections of a suitable vector-ordered group (and Boolean group). *G. Birkhoff.*

Jónsson, Bjarni, and Tarski, Alfred. Boolean algebras with operators. I. Amer. J. Math. **73**, 891–939 (1951).

The representation theorem for boolean algebras (herein b.a.'s) is stated in the following form: Every b.a. B has a "perfect" extension A (essentially unique), where A is complete and atomistic, where any covering of the universal element of A by elements from B contains a finite subcovering, and where, for any distinct atoms u and v of A, there exists an element of B containing u but not v. Then B is called a "regular" subalgebra of A. An element of A is called "closed" if it is the meet of all elements of B that contain it. Let the concepts of inclusion and closure be extended to the product A^m in the obvious fashion. Now, for any function f: B^m into B, define f^+: A^m into A, by $f^+(x)=\sum_y\prod_z f(z)$ for $y\subset x$, y closed, and $y\subset z$, z in B^m. If f is additive (in each argument, with $+$ as union), then f^+ is a completely additive extension of f. Moreover, it is shown that any functional identity among additive functions of B carries over to their extensions in A.

By a "boolean algebra with operators",

$$(B, +, 0, \cdot, 1, f_0, f_1, \cdots),$$

is meant a b.a. B with certain additive functions $f_0, f_1, \cdots$. Evidently, in the indicated sense, every b.a. with operators B has a unique representation as a regular subalgebra of its perfect extension A, with all functional identities of B carrying over to A. Let the b.a. A be identified with the algebra of all subsets of a set U, and suppose further that the operators f_i, and hence the f_i^+, vanish when any argument is 0. Then each such f^+: A^n into A, may be correlated with a relation $R\subset U^{n+1}$ by

$$f^+(x_1, \cdots, x_n)=\{\xi_{n+1}|R(\xi_1, \cdots, \xi_{n+1}), \xi_1\epsilon x_1, \cdots, \xi_n\epsilon x_n\}.$$

Thus B is represented as a regular subalgebra of the "complex algebra" of the abstract algebra $(U, R_0, R_1, \cdots)$ over U with relations $R_0, R_1, \cdots$. For B an abstract closure algebra, this yields a representation in which the closure operator f^+ is correlated with a transitive and reflexive relation R. On the space U, it may fail that $f^+(x)=x$ whenever x is a single point; but on the other hand, f^+ is completely additive.

"Cylindric algebras" are abstracted from b.a.'s over a product space $U_0\times U_1$, with operators C_0, C_1 defined by

$$C_ix=\{(\eta_0, \eta_1)|\exists(\xi_0, \xi_1)\epsilon x, \xi_i=\eta_i\}.$$

A representation is obtained as a regular subalgebra of the complex algebra over U with two equivalence relations R_0, R_1, whose relative product is the universal relation.

This sketch of the central results necessarily neglects many technical details of intrinsic interest, as well as the general aspect under which the subject is viewed.

R. C. Lyndon (Princeton, N. J.).

Foster, Alfred L. Ring-logics and p-rings. Univ. California Publ. Math. (N.S.) **1**, 385–395 (1951).

Continuant des articles antérieurs [Amer. J. Math. **72**, 101–123 (1950); Acta Math. **84**, 231–261 (1951); ces Rev. **11**, 414; **12**, 584], l'auteur établit des identités dans un p-anneau (anneau où $px=0$ et $x^p=x$, p premier), en faisant intervenir de nouvelles opérations telles que $x^\wedge=1+x$, $a\times'b=a+b+ab$, etc. *J. Dieudonné* (Nancy).

Herstein, I. N. A generalization of a theorem of Jacobson. Amer. J. Math. **73**, 756–762 (1951).

Theorem: If R is a ring with center C, and if $x^n-x\epsilon C$ for all x in R, n a fixed integer larger than 1, then R is commutative. This result is proved first for division rings and then primitive rings, from which it follows for rings which are semisimple in the sense of Jacobson. This leads to the location of the commutators and radical of R. The theorem is then proved for subdirectly irreducible rings, from which it follows for a general ring R. The author conjectures that, in the hypothesis, the fixed integer n may be replaced by $n(x)$, as is the case in the theorem of Jacobson to which the title refers. *R. D. Schafer* (Philadelphia, Pa.).

√*** Jacobson, N. Representation theory for Jordan rings.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 37–43. Amer. Math. Soc., Providence, R. I., 1952.

This paper, an invited address at the Algebra Conference of the 1950 International Congress of Mathematicians, gives an account of the author's theory of representations of (abstract) Jordan rings and the results of the author and C. E. Rickart on Jordan homomorphisms of (associative) rings. Detailed expositions have subsequently appeared and been reviewed in these Reviews [Jacobson, Trans. Amer. Math. Soc. **70**, 509–530 (1951); Jacobson and Rickart, ibid. **69**, 479–502 (1950); these Rev. **12**, 797, 387].

R. D. Schafer (Philadelphia, Pa.).

Falk, Gottfried. Über Ringe mit Poisson-Klammern. Math. Ann. **123**, 379–391 (1951).

The author first introduces the free polynomial ring $\mathfrak{F}$ with unit element which is generated by n noncommutative indeterminates over the field of real numbers. If $n=2s$ and $(\varphi, \psi)=\varphi\psi-\psi\varphi$, one homomorphic image of $\mathfrak{F}$ is the Heisenberg ring $\mathfrak{F}_H$ whose elements consist of polynomials in $p_1, \cdots, p_s, q_1, \cdots, q_s$, where these are subject to the following defining relations: $(q_i, q_k)=(p_i, p_k)=(p_i, q_k)=0$ for $i\neq k$; $(p_i, q_i)=(p_k, q_k)$ for $i, k=1, 2, \cdots, s$. It follows that (p_i, q_i) is commutative with all elements of $\mathfrak{F}_H$. Next consider a ring $\mathfrak{G}$ with unit element which contains the real field in its center and which is generated by elements $p_1, \cdots, p_s$, $q_1, \cdots, q_s$. Furthermore, it is assumed that a "bracket expression" $[\varphi, \psi]$ is defined for elements φ, ψ of $\mathfrak{G}$ and that this bracket expression satisfies a prescribed system of

postulates which are suggested by properties of the Poisson brackets. It is then shown that $\mathfrak{G}$ is either a commutative ring or a Heisenberg ring. Differential ideals in $\mathfrak{F}$ (using Hausdorff differentiation) are also briefly considered. There are some contacts with the literature which the author has overlooked. See, for example, a paper by the reviewer [Trans. Amer. Math. Soc. **39**, 101–116 (1936)] and other references there given. *N. H. McCoy.*

Schwarz, Ludwig. Bemerkung zu der Note von G. Herglotz "Eine Formel der formalen Operatorenrechnung." Math. Ann. **123**, 406–410 (1951).

A generalization of a result established by Herglotz [Math. Ann. **122**, 14–15 (1950); these Rev. **12**, 155].

N. H. McCoy (Northampton, Mass.).

√ ***Dieudonné, Jean. Les idéaux minimaux dans les anneaux associatifs.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 44–48. Amer. Math. Soc., Providence, R. I., 1952.

The methods introduced by the author in a previous paper [Bull. Soc. Math. France **70**, 46–75 (1942); these Rev. **6**, 144] are applied here in order to obtain more general results. We recall that the right (left) socle of a ring A is defined as the sum S of all minimal right (left) ideals of A, and is a two sided ideal (loc. cit.). If A has no nonzero nilpotent ideals (this condition will be assumed henceforth), right and left socles coincide and we refer to S as the socle of A. The socle S is the direct sum of simple rings M_α. Denote by T the right annihilator of S in A. Then T is also the left annihilator of S; T is a two sided ideal in A, and $S\cap T=0$. The author introduces a topology in A by prescribing as neighborhoods of 0 the right annihilators in A of the finite subsets of $S+T$ and shows that in respect to this topology A becomes a topological ring. Denote by $\bar{A}$ the completion of A, then the right annihilators in $\bar{A}$ of the finite subsets of $S+T$ constitute a complete set of neighborhoods of 0 in $\bar{A}$. If $\bar{S}$ and $\bar{T}$ are the closures of S and T in $\bar{A}$, and T_1 is the annihilator (right and left) of $\bar{S}$ in $\bar{A}$, then $T_1\supseteq\bar{T}$ and $\bar{A}$ is the direct sum of $\bar{S}$ and T_1. The ideal $\bar{S}$ contains all one sided minimal ideals of $\bar{A}$ and is isomorphic with the complete direct sum of the closures $\bar{M}_\alpha$ of the above mentioned simple rings M_α. As to the structure of M_α and $\bar{M}_\alpha$, this was described by the author in detail in terms of certain vector spaces and their duals in the paper cited above. The author shows further that if A is semi-simple (in the sense of Jacobson) then so is $\bar{A}$. He finally considers the case where $T=0$, which implies that $T_1=0$, i.e. $\bar{A}=\bar{S}$ and $S\subseteq A\subseteq\bar{S}$. This result yields a new proof of P. Jaffard's theorem [C. R. Acad. Sci. Paris **229**, 805–806 (1949); these Rev. **11**, 311] which characterizes the rings with a nonzero socle whose annihilator is 0 and in case A is primitive also contains a result due to Jacobson [Ann. of Math. (2) **48**, 8–21 (1947); these Rev. **8**, 433] concerning the characterization of primitive rings that contain minimal one-sided ideals. The author remarks that in general $T_1\neq\bar{T}$ and gives an example to this effect. *J. Levitski* (Jerusalem).

Artin, Emil, and Tate, John T. A note on finite ring extensions. J. Math. Soc. Japan **3**, 74–77 (1951).

Let R and S be two commutative rings, $R\subseteq S$. Then S is called a module-finite extension of R, if it is an R-module with a finite set of generators, that is, if there exists a finite number of elements of S such that every element of S can be represented as a linear combination of them with coefficients in R. On the other hand, S is said to be a ring-finite extension of R, if it can be written in the form $S=R[\xi_1, \xi_2, \cdots, \xi_n]$. The following theorem is proved. Let R be a Noetherian ring with unit element, let S be a ring-finite extension, and let T be an intermediate ring, $R\subseteq T\subseteq S$, such that S is a module-finite extension of T. Then T is a ring-finite extension of R. As an application, the following theorem of Zariski [Bull. Amer. Math. Soc. **53**, 362–368 (1947); these Rev. **8**, 499] is obtained. If a ring-finite extension of a field is a field, then it is algebraic and hence module-finite. As shown by Zariski, the Nullstellensatz is an immediate consequence of this result. It is further shown that a Noetherian integral domain R with a unit element has ring-finite extensions which are fields if and only if the quotient field F of R is a ring-finite extension of R. The ring-finite extension fields of R are then exactly the module-finite extension fields of F. The condition that F is a ring-finite extension of R is equivalent to each of the following conditions. I. There exists an element $a\neq 0$ of R which is contained in all proper prime ideals of R. II. There exists only a finite-number of minimal prime ideals of R. III. There exists only a finite number of prime ideals of R, and every one of them is maximal. *R. Brauer.*

Goldman, Oscar. Hilbert rings and the Hilbert Nullstellensatz. Math. Z. **54**, 136–140 (1951).

Hilbert's Nullstellensatz, in a polynomial ring over a field, can be re-interpreted as asserting that every prime ideal is an intersection of maximal ideals. Thus the author calls a (commutative) ring (with unit element) a Hilbert ring if it possesses this property. It is proved, among others, that R is a Hilbert ring if and only if the polynomial ring $R[x]$ is such. The proof is a skilful combination of the properties of prime ideals, maximal ideals and polynomials. The usual Hilbert Nullstellensatz follows naturally as a corollary. Some characteristic properties of Hilbert rings are obtained. One is that every non-maximal prime ideal is the intersection of properly larger prime ideals. Further, R is a Hilbert ring if and only if every maximal ideal in $R[x]$ contracts in R to a maximal ideal. A corollary is that if k is a field and M is a maximal ideal in $k[x_1, \cdots, x_n]$ then $k[x_1, \cdots, x_n]/M$ is algebraic over k; for this cf. also Zariski [Bull. Amer. Math. Soc. **53**, 362–368 (1947); these Rev. **8**, 499] and the paper of Artin and Tate reviewed above. *T. Nakayama.*

Almeida Costa, A. Über Kontraktions- und Vernichtungsideale in der allgemeinen Modulntheorie. Centro Estudos Mat. Fac. Ci. Pôrto. Publ. no. **23** = Univ. Lisboa. Revista Fac. Ci. A. Ci. Mat. (2) **1**, 297–343 (1951).

This paper is a continuation of an earlier one [same Estudos Publ. no. **20** = Anais Fac. Ci. Porto **33** (1948); these Rev. **12**, 9]. Let M be a module, R its ring of endomorphisms. With any submodule N we may associate two ideals in R: the right ideal annihilating N, and the left ideal of all endomorphisms with range in N. The author is concerned with the interplay between these ideals and the structure of R (its chain conditions, reducibility and idempotents).

I. Kaplansky (Chicago, Ill.).

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Almeida Costa, A. On contraction and annihilating ideals in the general theory of modules. Anais Fac. Ci. Porto **35**, 82 pp. (1951). (Portuguese)

The first nine sections are, except for certain small corrections, the same as in the paper reviewed above. The tenth, on the Krull-Schmidt theorem has been amplified.

The remaining sections are new material on infinite direct sums, primitive rings, semi-simple rings and modules.
I. Kaplansky (Chicago, Ill.).

Almeida Costa, A. On rings of endomorphisms. Gaz. Mat., Lisboa **12**, no. 50, 17–21 (1951). (Portuguese)

The author amplifies portions of the two preceding papers.
I. Kaplansky (Chicago, Ill.).

Krishnaswami Iyengar, S., Sreenivasa Rao, K. N., and Srinivasan, C. Examples on non-commutative algebras. Half-Yearly J. Mysore Univ. Sect. B., N.S. **9**, 51–56 (1948).

The authors give explicit determinations of the decompositions into minimal left ideals and the corresponding representations of: 1) the Duffin-Kemmer algebra with 2 generators; 2) the ξ-algebra [cf. Madhavarao, Thiruvenkatachar, and Venkatachaliengar, Proc. Roy. Soc. London. Ser. A. **187**, 385–397 (1946); these Rev. **8**, 194] for the case of 2 generators and spin 3/2. *I. E. Segal* (Princeton, N. J.).

Harish-Chandra. On some applications of the universal enveloping algebra of a semisimple Lie algebra. Trans. Amer. Math. Soc. **70**, 28–96 (1951).

Cet article étant exceptionnellement important par l'originalité des résultats qu'il contient, on espère qu'on voudra bien excuser le rapporteur de publier une analyse exceptionnellement longue.

I. Soit **L** une algèbre de Lie semi-simple sur le corps complexe (en fait, sur un corps algébriquement clos de caractéristique 0). Soient $\langle X, Y\rangle$ la forme de Killing de **L** et **h** une sous-algèbre de Cartan de **L**; on sait que $\langle X, Y\rangle$ n'est pas dégénérée sur **h** de sorte qu'à toute forme linéaire w sur **h** correspond un $H_w'\epsilon\mathbf{h}$ tel que $w(H)=\langle H_w', H\rangle$ pour tout $H\epsilon\mathbf{h}$. En particulier, soit α une racine de **L** par rapport à **h** (i.e. une forme linéaire sur **h** telle qu'il existe $X_\alpha\epsilon\mathbf{L}$ non nul avec $[H, X_\alpha]=\alpha(H)\cdot X_\alpha$ pour tout $H\epsilon\mathbf{h}$); on sait que $\alpha(H_\alpha')$ est un nombre rationnel >0; posons $H_\alpha=2H_\alpha'/\alpha(H_\alpha')$ et introduisons dans le dual $\mathbf{h}'$ de l'espace vectoriel **h** l'opérateur de symétrie $w\rightarrow s_\alpha(w)=w-w(H_\alpha)\cdot\alpha$; alors les s_α engendrent un groupe fini W (groupe de Weyl) qui permute entre elles les racines. En considérant un domaine fondamental de W dans $\mathbf{h}'$, on montre qu'on peut trouver l racines α_i (l=dimension de **h**) telles que toute racine soit combinaison linéaires à coefficients entiers et tous de même signe des α_i; on dit que les α_i forment un système fondamental de racines de **L** par rapport à **h**. Si l'on pose $s_i=s_{\alpha_i}$ on a des relations $s_i(\alpha_j)=\alpha_j+a_{ij}\alpha_i$, où les $a_{ij}=-\alpha_i(H_{\alpha_j})$ sont des entiers qui déterminent W et possèdent les propriétés suivantes: $a_{ii}=-2$; $a_{ij}\geqq 0$ pour $i\neq j$; $a_{ij}=a_{ji}$; det $(a_{ij})\neq 0$. Ces a_{ij} sont les "entiers de Cartan" de **L** (par rapport au système fondamental considéré).

Soit d'autre part $X\rightarrow\rho(X)$ une représentation linéaire de **L** dans un espace vectoriel V de dimension finie; disons qu'une forme linéaire $\lambda\epsilon\mathbf{h}'$ est un poids de la représentation si le sous-espace V_λ des $\mathbf{x}\epsilon V$ tels que $\rho(H)\mathbf{x}=\lambda(H)\cdot\mathbf{x}$ n'est pas nul; on sait que V est somme directe des divers V_λ (i.e. la représentation $H\rightarrow\rho(H)$ est complètement réductible), que tout poids est de la forme $\lambda=\sum c_i\alpha_i$ où les c_i sont rationnels, et que les $\lambda(H_i)$, où $H_i=H_{\alpha_i}$, sont des nombres entiers. Si l'on introduit dans l'ensemble des poids possibles la relation d'ordre lexicographique par rapport à la base (α_i) on peut parler du plus haut poids de ρ; pour celui-ci, les $\lambda(H_i)$ sont des entiers $\geqq 0$.

Ces propriétés classiques étant rappelées, la première partie de l'article consiste à montrer que toutes ces propriétés ont des réciproques, à savoir: 1) pour tout système d'entiers positifs λ_i il existe une et une seule représentation irréductible de **L** dont le plus haut poids λ est donné par $\lambda(H_i)=\lambda_i$; 2) tout système d'entiers de Cartan correspond à une et une seule algèbre de Lie semi-simple.

Ces propriétés ne sont pas nouvelles (contrairement à celles que nous analyserons plus loin), mais l'auteur en donne, pour la première fois, une démonstration directe et purement algébrique (démonstration trouvée en même temps par C. Chevalley). Les a_{ij} étant donnés, on construit une algèbre associative infinie **A** par la condition qu'elle possède $3l$ générateurs H_i, X_i, Y_i reliés par les seules relations: $[H_i, H_j]=0$; $[X_i, Y_i]=H_i$; $[X_i, Y_j]=0$, $i\neq j$; $[H_i, X_j]=-a_{ij}X_j$; $[H_i, Y_j]=a_{ij}Y_j$. (Evidemment **A** est l'algèbre universelle enveloppante de l'algèbre de Lie **L** qu'il s'agit de construire.) Etant donnée une forme linéaire λ sur le sous-espace **h** engendré par les H_i, l'auteur montre à priori qu'il existe un seul idéal à gauche maximal M_λ de **A** contenant les éléments $H_i-\lambda(H_i)$ et les X_i; de plus si les $\lambda(H_i)$ sont entiers, alors l'espace $\mathbf{A}/M_\lambda$ est de dimension finie, et en considérant la représentation naturelle de **A** dans cet espace, laquelle est irréductible et de dimension finie, on voit que λ est le plus haut poids de cette représentation, qui est parfaitement déterminée par λ en raison de l'unicité de l'idéal M_λ. Quant à l'algèbre de Lie **L** admettant le système donné d'entiers de Cartan, on l'obtient en construisant la somme directe π des représentations irréductibles π_i dont les plus hauts poids λ_i sont donnés par $\lambda_i(H_j)=\delta_{ij}$; **L** est alors la sous-algèbre de Lie de $\pi(\mathbf{A})$ engendrée par les éléments $\pi(H_i)$, $\pi(X_i)$ et $\pi(Y_i)$.

L'auteur obtient aussi divers autres résultats, notamment des limitations sur la dimension de la représentation irréductible de plus haut poids donné (mais non les formules exactes, que jusqu'à nouvel ordre on ne peut établir que par les méthodes "transcendantes" d'H. Weyl).

II. La seconde partie de l'article est inspirée du problème suivant: soient G un groupe semi-simple complexe et K un sous-groupe compact maximal de G; comment les représentations irréductibles (de dimension $\leqq+\infty$) de G se décomposent-elles lorsqu'on les restreint à K, et comment peut-on les classifier? Le problème traité par l'auteur est une transposition purement algébrique du précédent.

Soit **L** la complexification de l'algèbre de Lie réelle de G; soit $\mathbf{L}_K$ l'analogue de **L** pour K; $\mathbf{L}_K$ s'identifie à une sous-algèbre (semi-simple complexe) de **L**; le problème est alors le suivant: comment les représentations irréductibles de **L** se décomposent-elles lorsqu'on les restreint à $\mathbf{L}_K$? Pour traiter ce problème il faut faire des hypothèses sur ces représentations (hypothèses évidemment réalisées lorsqu'on est parti de représentations du groupe G correspondant à **L**), à savoir: 1) la représentation ρ donnée, restreinte à $\mathbf{L}_K$, se décompose complètement en représentations irréductibles de dimensions finies; 2) **B** désignant l'algèbre enveloppante universelle de **L** et **Z** le centre de **B**, $\rho(z)$ est un scalaire pour tout $z\epsilon\mathbf{Z}$. Les résultats obtenus par l'auteur sont alors que: (a) ρ ne contient qu'un nombre fini de fois une représentation irréductible donnée $\mathfrak{d}$ de $\mathbf{L}_K$; (b) appelons "caractère" de ρ la forme linéaire $\chi_\rho(z)$ sur **Z** telle que $\rho(z)=\chi_\rho(z)\cdot 1$; alors il n'existe qu'un nombre fini de représentations irréductibles ρ admettant un caractère donné et faisant intervenir une représentation irréductible donnée $\mathfrak{d}$ de $\mathbf{L}_K$.

Les démonstrations sont fort ingénieuses, et nous allons indiquer comment on parvient à (a). Puisque G se déduit de K par complexification, il existe une application linéaire Γ de **L** sur **L** vérifiant $[\Gamma(X), Y]=\Gamma([X, Y])$, $\Gamma^2=-1$, et

telle que $\mathbf{L}$ soit (en tant qu'espace vectoriel) somme directe de $\mathbf{L}_K$ et de $\Gamma(\mathbf{L}_K)$ (si l'on munit l'algèbre de Lie réelle de G de sa structure complexe naturelle, Γ se réduit sur celle-ci à la multiplication par $i=(-1)^{\frac{1}{2}}$). Posant $\gamma^+=\frac{1}{2}(1+i\cdot\Gamma)$, $\gamma^-=\frac{1}{2}(1-i\cdot\Gamma)$, $\mathfrak{g}^+=\gamma^+(\mathbf{L}_K)$, $\mathfrak{g}^-=\gamma^-(\mathbf{L}_K)$ il est immédiat de constater que γ^+ et γ^- sont des isomorphismes de $\mathbf{L}_K$ sur des idéaux $\mathfrak{g}^+$ et $\mathfrak{g}^-$ de $\mathbf{L}$, idéaux dont la somme directe est $\mathbf{L}$. On vérifie aussi que l'espace $\mathbf{L}$ est somme directe des sous-espaces $\mathfrak{g}^+$ et $\mathbf{L}_K$. Soient $\mathbf{A}^+$ et $\mathbf{X}$ les algèbres enveloppantes de $\mathfrak{g}^+$ et de $\mathbf{L}_K$; il est facile de voir que $\sum a_i\otimes x_i\to\sum a_i x_i$ applique $\mathbf{A}^+\otimes\mathbf{X}$ sur $\mathbf{B}$.

Soit alors ρ une représentation irréductible de $\mathbf{L}$ sur $\mathfrak{H}$; pour $\mathfrak{d}$ donnée, soit $\mathfrak{H}(\mathfrak{d})$ le sous-espace engendré par les $\mathfrak{d}$-composantes de la représentation ρ restreinte à $\mathbf{L}_K$; soient $X\to(\sigma_\lambda{}^\mu(X))$ une réalisation matricielle de $\mathfrak{d}$, et $\mathbf{e}_\lambda$ des vecteurs de $\mathfrak{H}(\mathfrak{d})$ tels que $\rho(X)\mathbf{e}_\lambda=\sigma_\lambda{}^\mu(X)\mathbf{e}_\mu$ (convention d'Einstein); soit $\mathfrak{B}$ l'ensemble des combinaisons linéaires des $\mathbf{e}_\lambda$, et posons $T=\mathbf{A}^+\otimes\mathfrak{B}$. Si l'on observe que, en tant que sous-algèbre de $\mathbf{B}$, $\mathbf{A}^+$ est invariante par les opérateurs $a\to[X, a]$ et que d'autre part $\mathfrak{H}$ est l'ensembles des vecteurs $\rho(a^\lambda)\mathbf{e}_\lambda$, $a^\lambda\in\mathbf{A}^+$ (irréductibilité!) on voit que l'application linéaire θ: $\mathbf{a}\otimes\mathbf{x}\to\rho(\mathbf{a})\mathbf{x}$ de $\mathbf{A}^+\otimes\mathfrak{B}=T$ sur $\mathfrak{H}$ transforme en la représentation ρ de $\mathbf{L}_K$ la représentation ν de $\mathbf{L}_K$ dans T donnée par

$$(*)\qquad \nu(X)(a^\lambda\otimes\mathbf{e}_\lambda)=[X, a^\lambda]\otimes\mathbf{e}_\lambda+a^\lambda\otimes\sigma_\lambda{}^\mu(X)\mathbf{e}_\mu;$$

il est visible que θ applique $T(\mathfrak{d})$ (défini de façon évidente à partir de la représentation (*)) sur $\mathfrak{H}(\mathfrak{d})$, ceci parce que la représentation ν de $\mathbf{L}_K$ dans T est, comme on le voit facilement, complètement réductible.

D'autre part, soit $\mathbf{C}^+$ le centre de $\mathbf{A}^+$, évidemment contenu dans $\mathbf{Z}$; les $\rho(c)$, $c\in\mathbf{C}^+$, sont des scalaires; mais $\rho(c)$ se déduit par θ de l'opérateur $a\otimes\mathbf{x}\to(ca)\otimes\mathbf{x}$ sur T; ces opérateurs conservant évidemment $T(\mathfrak{d})$, on est ramené, pour prouver que $\mathfrak{H}(\mathfrak{d})$ est de dimension finie, à faire voir que $T(\mathfrak{d})$ est un $\mathbf{C}^+$-module de type fini. Pour cela on identifie $\mathbf{A}^+$ à $\mathbf{X}$ et on choisit une base X_i de $\mathbf{L}_K$; tout $a\in\mathbf{A}^+$ s'écrit alors comme polynôme non commutatif mais symétrique par rapport aux X_i; soit $a(x)$ le polynôme commutatif correspondant par rapport à des variables x_i et posons $[X, X_i]=\alpha_i{}^j(X)\cdot X_j$. Si des éléments $a_\lambda{}^\rho\otimes\mathbf{e}_\rho$ de T vérifient $\nu(X)(a_\lambda{}^\rho\otimes\mathbf{e}_\rho)=\sigma_\lambda{}^\mu(X)a_\mu{}^\rho\otimes\mathbf{e}_\rho$, on doit avoir d'après (*) les relations

$$(**)\qquad [X, a_\lambda{}^\rho]+\sigma_\mu{}^\rho(X)a_\lambda{}^\mu-\sigma_\lambda{}^\mu(X)a_\mu{}^\rho=0;$$

mais choisissons deux nouvelles séries de variables y_ρ, z^λ et formons le polynôme $g(x, y, z)=a_\lambda{}^\rho(x)y_\rho z^\lambda$; alors (**) indique que g est un invariant de la représentation de $\mathbf{L}_K$ dans $C[x, y, z]$ qui, à $X\in\mathbf{L}_K$, associe la dérivation transformant x_i en $\alpha_i{}^j(X)x_j$, y_ρ en $\sigma_\rho{}^\lambda(X)y_\lambda$ et z^λ en $-\sigma_\mu{}^\lambda(X)z^\mu$; appliquant le théorème connu de Hilbert, on voit que g s'exprime rationnellement au moyen d'un nombre fini d'invariants $J_k(x, y, z)$, $1\leqq k\leqq N$. Comme g est homogène de degré 1 en les y_ρ et les z^λ, on peut supposer que les J_k qui interviennent sont de degré $\leqq 1$ en les y_ρ et les z^λ; mettant en évidence la partie homogène de J_k, soit $f_k(x, y, z)$, il vient $g(x, y, z)=\sum p_k(x)\cdot f_k(x, y, z)$ où les $p_k(x)$ sont des invariants de la représentation adjointe de $\mathbf{L}_K$ et dépendent de g, tandis que les f_k n'en dépendent pas. Posant $f_k(x, y, z)=u^\rho{}_{k,\lambda}(x)y_\rho z^\lambda$ et désignant par $u^\rho{}_{k,\lambda}$ les éléments de $\mathbf{A}^+$ associés aux polynômes $u^\rho{}_{k,\lambda}(x)$ (écrits sous forme symétrique) on vérifie facilement que: 1) les $u^\rho{}_{k,\lambda}\otimes\mathbf{e}_\rho$ sont dans $T(\mathfrak{d})$; 2) ils engendrent le $\mathbf{C}^+$-module $T(\mathfrak{d})$.

Bien entendu, on peut, au lieu de considérer précisément $T(\mathfrak{d})$, étudier-avec les mêmes résultats—les $T(\mathfrak{d}')$ où $\mathfrak{d}'$ est quelconque.

III. Soient $\mathbf{L}$ une algèbre de Lie semi-simple complexe, $\mathbf{B}$ l'algèbre enveloppante correspondante, $\mathbf{Z}$ le centre de $\mathbf{B}$. Pour des raisons évidentes, on appelle caractère de $\mathbf{B}$ toute forme linéaire $\chi(b)$ sur $\mathbf{B}$ telle que $\chi(b'b'')=\chi(b''b')$ et qui, sur $\mathbf{Z}$, est un homomorphisme sur le corps complexe. L'auteur détermine tous ces caractères et obtient les résultats suivants.

Tout d'abord, considérons la représentation adjointe de $\mathbf{L}$ dans $\mathbf{B}$ qui, à $X\in\mathbf{L}$, associe l'opérateur $b\to[X, b]$; il est clair qu'elle est complètement réductible, donc que $\mathbf{B}$ est somme directe de $\mathbf{Z}$ et du sous-espace engendré par les $[X, b]$; par suite, il suffit de déterminer χ sur $\mathbf{Z}$, ce qui revient à déterminer les homomorphismes de l'algèbre commutative $\mathbf{Z}$ sur le corps complexe. Pour cela il faut avoir des renseignements précis sur la structure de $\mathbf{Z}$; l'idée essentielle de l'auteur est d'utiliser le fait qu'on connait déja un système complet de tels homomorphismes, savoir les caractères $\chi_\rho(b)=\mathrm{Sp}\,(\rho(b))$ des représentations irréductibles de dimension finie de $\mathbf{L}$ (Sp est la trace usuelle normalisée par $\mathrm{Sp}\,(1)=1$), ce qui va permettre, comme on le verra, de réaliser $\mathbf{Z}$ comme algèbre de polynômes (commutatifs).

Pour cela choisissons comme dans la partie I une base H_i de $\mathbf{h}$, désignons par α_j $(1\leqq j\leqq r)$ les diverses racines >0 de $\mathbf{L}$, et soit X_j (resp. Y_j) un élément de $\mathbf{L}$ appartenant à la racine α_j (resp. $-\alpha_j$), avec les normalisations usuelles. Soient $\mathbf{H}$ la sous-algèbre de $\mathbf{B}$ engendrée par les H_i et $\mathbf{P}$ l'idéal à gauche engendré par les X_i; on va montrer qu'il existe un isomorphisme $z\to z'$ de $\mathbf{Z}$ dans $\mathbf{H}$ tel que $z'\equiv z$ (mod. $\mathbf{P}$) pour tout $z\in\mathbf{Z}$. Tout d'abord, tout $z\in\mathbf{Z}$ (et même tout élément de $\mathbf{B}$) s'exprime linéairement au moyen des monômes

$$e(q, m, p)=Y_1{}^{q_1}\cdots Y_r{}^{q_r}H_1{}^{m_1}\cdots H_l{}^{m_l}X_1{}^{p_1}\cdots X_r{}^{p_r},$$

lesquels, comme on le voit aisément, vérifient

$$[H, e(q, m, p)]=w(H)e(q, m, p)$$

où $w(H)=\sum p_i\alpha_i(H)-\sum q_i\alpha_i(H)$. Donc (en raison de $[H, z]=0$) tout $z\in\mathbf{Z}$ s'exprime linéairement à l'aide des $e(p, m, p)$; dans cette expression de z désignons par z' l'ensemble des termes pour lesquels tous les p_i sont nuls; il est clair que $z'\in\mathbf{H}$ et que $z\equiv z'$ (mod. $\mathbf{P}$). Si nous montrons que cette dernière relation détermine entièrement z', il sera évident (puisque $\mathbf{Z}$ commute avec $\mathbf{P}$) que $z\to z'$ est un homomorphisme; nous allons prouver en même temps que cette application est biunivoque, ce qui prouvera toutes nos assertions. Pour cela, désignons par ρ_λ la représentation irréductible (de dimension finie) de plus haut poids λ et par $\mathbf{a}$ un vecteur de poids λ dans cette représentation; il est bien connu que $\mathbf{a}$ est annulé par les $\rho_\lambda(X_i)$, donc par $\mathbf{P}$, d'où résulte que $\rho_\lambda(z)\mathbf{a}=\rho_\lambda(z')\mathbf{a}$; posant $z'=f_z(H_1, \cdots, H_l)$ où f_z est un polynôme, cela s'écrit $f_z(\lambda)\mathbf{a}=\chi_\lambda(z)\mathbf{a}$ où $f_z(\lambda)$ est la valeur de f_z au point de coordonnées $\lambda(H_i)$ et où χ_λ est le caractère de ρ_λ. Donc on a la relation

$$(***)\qquad f_z(\lambda)=\chi_\lambda(z)$$

ce qui prouve: 1) l'unicité de f_z, puisqu'on le connait pour toutes les valeurs entières positives des variables; 2) le fait que $f_z=0$ implique $z=0$, car les représentations de dimension finie forment, comme on le sait, un système complet de représentations de $\mathbf{B}$, donc de $\mathbf{Z}$.

D'après (***) on peut identifier $\mathbf{Z}$ à l'algèbre des polynômes f_z sur l'espace $\mathbf{h}'$ dual de $\mathbf{h}$. Ceci permet immédiatement de former des caractères de $\mathbf{Z}$: on choisit, arbitrairement cette fois, une forme linéaire λ sur $\mathbf{h}$, et on pose $\chi_\lambda(z)=f_z(\lambda)$. Il se trouve que cette méthode de "prolonge-

ment algébrique" conduit à tous les caractères de **Z**. Pour le démontrer, l'auteur est obligé de faire des calculs fort pénibles (d'autant plus pénibles que l'auteur se croit obligé d'employer des notations telles que

$$\exp((-1)^{\frac{1}{2}}\alpha(H_i))-\exp(-(-1)^{\frac{1}{2}}\alpha(H_i))$$

là où il serait si simple d'écrire $2i\cdot\sin\alpha(H_i)\cdots$) que le rapporteur est tout aussi incapable de simplifier. La méthode consiste à montrer, à l'aide des formules explicites de H. Weyl donnant les caractères (au sens classique), que, si l'on pose $\rho=\frac{1}{2}$-somme des racines positives de **L**, l'algèbre des f_s s'identifie, par la translation $\lambda\rightarrow\rho+\lambda$ dans $\mathbf{h}'$, à l'algèbre **J** des invariants du groupe de Weyl W; une fois ce résultat acquis, tout est trivial; car W étant un groupe fini, l'algèbre de tous les polynômes sur $\mathbf{h}'$ est une extension de degré fini de **J**, de sorte que, pour employer le langage de la géométrie algébrique, toute "spécialisation" de **J** se prolonge à la dite algèbre—donc est de la forme $f\rightarrow f(\lambda)$. En outre, on voit aussi trivialement que $\chi_\lambda=\chi_\mu$ si et seulement si l'on a $\mu+\rho=s(\lambda+\rho)$ pour un $s\epsilon W$ (ce que l'auteur démontre de façon beaucoup trop compliquée; il suffit de montrer que les invariants d'un groupe compact W permettent de "séparer" les différents "orbites" de W, ce qui est clair par le théorème d'approximation de Weierstrass).

En ce qui concerne le fait que **Z** est isomorphe à **J** il a été démontré (mais non publié, évidemment) indépendamment de Harish-Chandra par H. Cartan, C. Chevalley, J. L. Koszul et A. Weil à propos de questions de topologie algébrique; leur méthode est probablement très différente; dans le cas où elle serait plus simple que celle de Harish-Chandra, il pourrait être intéressant d'en prendre connaissance.

Cette troisième partie se termine par l'étude du cas où **L** est du même genre que dans la partie précédente; cela revient à déterminer les caractères de l'algèbre enveloppante de $\mathbf{g}^+\oplus\mathbf{g}^-$, ce qui est évidemment facile maintenant.

IV. Nous arrivons à la dernière partie, dont le but est de passer des représentations de **L** à celles du groupe G (que l'auteur suppose simplement connexe). Pour cela l'auteur écrit $G=K\cdot S$ où S est un sous-groupe résoluble de G et, pour toute représentation de dimension un: $s\rightarrow\lambda(s)$ de S, considère la représentation induite correspondante de G; celle-ci, s'effectue de façon évidente dans l'espace de Hilbert $L^2(K)$, mais en général n'est pas irréductible. Toutefois, l'auteur montre qu'en restreignant ces représentations à des sous-espaces invariants bien choisis on obtient une famille de représentations de G qui, du point de vue "infinitésimal", possèdent tous les caractères possibles (mais ne sont toujours pas nécessairement irréductibles).

Par contre, l'auteur ne démontre pas que les résultats de la partie II s'étendent aux représentations du groupe G comme on pourrait s'y attendre; la raison en est évidemment que, pour les représentations de dimension infinie, le point de vue infinitésimal soulève des difficultés considérables (par exemple, il n'est pas évident que toute représentation possède des coefficients qui sont fonctions analytiques sur G); ces difficultés ont, depuis, été résolues par l'auteur [Proc. Nat. Acad. Sci. U. S. A. **37**, 170–173, 362–365, 366–369, 691–694 (1951); ces Rev. **13**, 106]; comme le montrera le rapporteur dans un article à paraitre des considérations d'analyse fonctionnelle permettent (en ce qui concerne l'extension des résultats de la partie II aux représentations de groupes) d'obtenir beaucoup plus simplement des résultats complets—ce qui ne veut d'ailleurs pas dire que les méthodes infinitésimales soient inutiles! *R. Godement.*

Theory of Groups

Tamura, Takayuki. Characterization of groupoids and semilattices by ideals in a semigroup. J. Sci. Gakugei Fac. Tokushima Univ. **1**, 37–44 (1950).

This note contains characterisations of two types of semigroup, viz. groupoids (by which the author understands semi-groups with a one-sided unit and in which division is possible on one side) and semi-lattices (or commutative idempotent semi-groups), by means of the structure of the lattices of ideals of these semi-groups. *D. Rees.*

Skolem, Th. Theory of divisibility in some commutative semi-groups. Norsk Mat. Tidsskr. **33**, 82–88 (1951).

The subject matter of this paper is the theory of divisibility in a commutative semi-group S with the following properties: a) S possesses an identity e; b) S satisfies the cancellation law; c) there exists an integer h such that, if a and b are two elements of S, a^h and b^h have a g.c.f. d, and, further, d^n is a g.c.f. of a^n, b^n for all n; d) the ascending chain condition holds for principal ideals in S; e) if a^n divides b^n then a divides b. Elements a and b of S are termed co-prime if the g.c.f. of a^h, b^h is e. Then a is termed semi-prime if a^h cannot be expressed in the form bc where b and c are co-prime and neither is a unit. Finally a is quasi-prime if it is semi-prime and is not composite. With these definitions, the main result of the paper is that, if a is any element of S, then a^h can be expressed as a product of powers of quasi-primes, and this expression is unique in the usual sense.

D. Rees (Cambridge, England).

Jaffard, Paul. Théorie axiomatique des groupes définis par des systèmes de générateurs. Bull. Sci. Math. (2) **75**, 114–128 (1951).

Conditions are found under which a factor of semi-group shall be a group. The word problem is discussed and proofs are given that the word problem is solvable for free groups and for free products of free groups with an amalgamated subgroup. *R. M. Thrall* (Ann Arbor, Mich.).

Higman, Graham. Almost free groups. Proc. London Math. Soc. (3) **1**, 284–290 (1951).

If the group G is the free product of its subgroups H and K, then H and K are called free factors of G. A group generated by $\mathfrak{a}$ elements ($\mathfrak{a}$ a cardinal number) is called an $\mathfrak{a}$-group. In particular, a finitely generated group is called an n-group. An $\mathfrak{a}$-subgroup of G is basic if it is a free factor of any subgroup containing it. A group G is said to be locally free if any n-subgroup of G is free. It is known that there exist locally free groups which are not free. The author says that a group G is countably free if all its countable subgroups are free. More generally, he calls a group G with order $\mathfrak{a}$ almost free if any $\mathfrak{b}$-subgroup with $\mathfrak{b}<\mathfrak{a}$ is free. He shows that G is countably free if and only if any properly ascending sequence of n-subgroups of G, no one of which is contained in a proper free factor of the following, must terminate after a finite number of steps. This condition is shown to be equivalent with the following: Every n-subgroup of G is contained in a basic n-subgroup of G. With the help of this result the author constructs a countably free group with cardinal number $\aleph_1$ which is not free. On the other hand he shows that if $\mathfrak{a}$ is the limit of a sequence $\mathfrak{a}_1, \mathfrak{a}_2, \cdots$ of type ω of infinite cardinals satisfying $2^{\mathfrak{a}_i}\leqq\mathfrak{a}_{i+1}$, then an almost free group of order $\mathfrak{a}$ is free.

J. Levitzki (Jerusalem).

Douglas, Jesse. On the basis theorem for finite Abelian groups. III. Proc. Nat. Acad. Sci. U. S. A. **37**, 611–614; errata, 716 (1951).

This third version of the author's proof for the basis theorem for finite primary Abelian groups is based on the following lemma: If a proper subgroup K of an Abelian p-group G possesses a basis, then there exists a subgroup H of G, containing K, which also possesses a basis. Closer inspection, however, reveals that the proof is completely equivalent to (although rather more complicated than) the classical constructive proof [see e.g. Burnside, Theory of groups of finite order, 2nd ed., Cambridge, 1911, p. 102]: select an element of maximal order, then one of maximal relative order; and from the two construct a second independent element; and continue the process until the group is exhausted. *K. A. Hirsch* (London).

Douglas, Jesse. On the invariants of finite Abelian groups. Proc. Nat. Acad. Sci. U. S. A. **37**, 672–677 (1951).

Let G be a finite primary Abelian group belonging to the prime number p, $I(x)$ the number of elements of G of order at most p^x, and $K(x)$ the number of elements of order p^x in any basis of G. Then $K(x)=2I(x)-I(x+1)-I(x-1)$. This formula is, of course, contained in the explicit formula for $I(x)$ [see e.g. Burnside, Theory of groups of finite order, 2nd ed., Cambridge, 1911, p. 104]. The author gives an independent proof and also some graphical interpretation of the functions $I(x)$ and $K(x)$. *K. A. Hirsch* (London).

Beaumont, Ross A., and Zuckerman, H. S. A characterization of the subgroups of the additive rationals. Pacific J. Math. **1**, 169–177 (1951).

The authors investigate the subgroups of the additive group R^+ of all rational numbers, i.e. the torsion-free abelian groups of rank one. A very simple invariant characterisation of these groups is given according to which one can decide in what case such a group is a subgroup of another, and whether two such groups are isomorphic or not. For two isomorphic subgroups of R^+ there are given all isomorphisms. In addition the authors determine the group of automorphisms and all homomorphic images of a subgroup of R^+.

The second part of the paper is devoted to the problem of determining all rings which have a given subgroup of R^+ as additive group. It is shown that such a ring is always isomorphic to a subring of the rational number field R or is a null ring. In addition, all subgroups S of R^+ are determined for which the only ring with S as additive group is the null ring. The group R^+ itself has the property that any ring with additive group R^+, which is not a null ring, is isomorphic to R. *T. Szele* (Debrecen).

Kazačkov, B. V. On theorems of Sylow type. Doklady Akad. Nauk SSSR (N.S.) **80**, 5–7 (1951). (Russian)

In the set of prime numbers, let Π be a non-empty subset and Θ its complement. A group is called of type $\Pi-S$ if all its Sylow Π-subgroups are conjugate; of type $\Pi-C$ if all its Sylow Π-subgroups are conjugate and solvable; of type $\Pi\Theta-C$ if all its Sylow Π-subgroups are conjugate and solvable and furthermore all its Sylow Θ-subgroups are conjugate. Let $\mathfrak{N}$ be a Π-solvable normal subgroup of the finite group $\mathfrak{G}$; if the factor-group $\mathfrak{G}/\mathfrak{N}$ is of type $\Pi-C$, so is $\mathfrak{G}$; similarly if $\mathfrak{G}/\mathfrak{N}$ is of type $\Pi\Theta-C$, so is $\mathfrak{G}$. If $\mathfrak{G}$ is a locally solvable group or if $\mathfrak{G}$ is a locally finite group each of whose finite subgroups is of type $\Pi-S$, then $\mathfrak{G}$ has the following property: if a Sylow Π-subgroup $\mathfrak{M}$ of $\mathfrak{G}$ has only a finite number of conjugates, then all Sylow Π-subgroups of $\mathfrak{G}$ are included among the conjugates of $\mathfrak{M}$. All Sylow Π-subgroups of a solvable group $\mathfrak{G}$ satisfying the minimal condition are conjugates of one another. *R. A. Good.*

Gluškov, V. M. On locally nilpotent groups without torsion. Doklady Akad. Nauk SSSR (N.S.) **80**, 157–160 (1951). (Russian)

For terminology, see an earlier paper [Doklady Akad. Nauk SSSR (N.S.) **74**, 885–888 (1950); these Rev. **12**, 477]. A locally nilpotent torsion-free group with the minimal condition for servant normal subgroups is called an M_0-group. In a locally nilpotent torsion-free group, finitely many subgroups of finite (Mal'cev) special rank generate a nilpotent subgroup of finite special rank, and the composite of arbitrarily many complete subgroups is a complete subgroup; the analogous results for periodic groups are not valid. For a torsion-free group the following three properties are equivalent: it is a nilpotent group of finite special rank; it is a locally nilpotent group satisfying the minimal condition for servant subgroups; it is a locally nilpotent group satisfying the maximal condition for servant normal subgroups. In order that a torsion-free group which is either an extension of a ZA-group by means of a group of finite special rank or else an extension of an M_0-group by means of a ZA-group by a ZA-group itself, it is necessary and sufficient that it be locally nilpotent. A locally nilpotent group $\mathfrak{G}$ which is the extension of an abelian normal subgroup $\mathfrak{A}$ by means of a group of finite special rank is a ZA-group, whose upper central series has length either k or $\omega+k$, where k is a natural number and ω the first limiting ordinal number, and in the latter case $\mathfrak{A}$ is contained in the ωth hypercenter of the group $\mathfrak{G}$. For an M_0-group the following three properties are equivalent: it is an extension of an abelian group by means of a group of finite special rank; one of its maximal abelian normal subgroups is contained in a hypercenter of index not exceeding ω; its upper central series has length less than the second limiting ordinal number. *R. A. Good.*

Kesava Menon, P. The invariants of finite transformation groups. I. Math. Student **18**, 100–107 (1950).

The author considers the rational invariants of an arbitrary finite group G of fractional linear transformations over an arbitrary field. He finds an invariant which defines G and uses this to find a fractional linear representation of K/G where K is any finite fractional linear group of which G is a normal subgroup. *E. R. Kolchin* (New York, N. Y.).

Hirsch, K. A. On infinite soluble groups. IV. J. London Math. Soc. **27**, 81–85 (1952).

The principal result of this note is the following theorem: If the maximum condition is satisfied by the subgroups of the group G, and if the derived series of G terminates with 1 after a finite number of steps, then 1 is the intersection of all subgroups of finite index of G. *R. Baer.*

Itô, Noboru. On the characters of soluble groups. Nagoya Math. J. **3**, 31–48 (1951).

Let p and q be a pair of primes satisfying the condition $q^s-1=p^t$. Let K be a holomorph of an abelian group of order q^s and of type $(q, \cdots, q)$ by a cyclic group of automorphisms of order p^t, and consider $H=K_1\times\cdots\times K_p$, $K_i\cong K$. For some fixed isomorphism $a\rightarrow a_i$ between K and K_i, $i=1, \cdots, p$, denote by Q the holomorph of H by a cyclic group $\{\prod_{e\neq a\in K}(a_1, \cdots, a_p)\}$ of automorphisms of

order p. Groups of the type of Q are said to be of the "first kind". One of the modular properties (in respect to p) of such groups is that they have no block of defect 0 and only one block of defect 1. The author proves the following theorem: If a soluble group G has no normal p-subgroups and no subgroups which ~~are~~ homomorphic images of a group of the first kind, then G has a character of defect zero. The author then introduces the so called groups of the "second kind" and derives a similar condition for the occurrence of positive defects. *J. Levitzki* (Jerusalem).

Itô, Noboru. On the degrees of irreducible representations of a finite group. Nagoya Math. J. 3, 5–6 (1951).

The author proves that the degree of any absolutely irreducible representation of a finite group divides the index of each of its (maximal) abelian normal subgroups.
J. Levitzki (Jerusalem).

√✱**Godement, Roger. Some unsolved problems in the theory of group representations.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 106–111. Amer. Math. Soc., Providence, R. I., 1952.

The author states some problems on unitary representations of locally compact topological groups and related topics. Some of these problems are due to the author, a number of them were suggested to him by others, but the form in which they now appear is due to Godement.
F. I. Mautner (Baltimore, Md.).

van Est, W. T. Dense imbeddings of Lie groups. Nederl. Akad. Wetensch. Proc. Ser. A. **54** = Indagationes Math. **13**, 321–328 (1951).

The problem is to determine under what conditions a continuous representation of one Lie group into another is closed. It is shown that a representation of a semisimple Lie group $\mathfrak{G}$ into a Lie group $\mathfrak{H}$ is closed if and only if the image of the center of $\mathfrak{G}$ is discrete. For example, a linear representation of a semisimple Lie group is always closed. A semisimple group with infinite center always admits a non-closed imbedding in a Lie group. If a Lie group $\mathfrak{G}$ is locally faithfully represented in a dense subgroup of a Lie group $\mathfrak{H}$, then the Lie algebra H decomposes into a direct sum of its center D and an ideal H_1 containing the image of the Lie algebra G. The author's results overlap those of Gotô [Math. Japonicae 1, 107–119 (1948); Nagoya Math. J. 1, 91–107 (1950); these Rev. 10, 681; 12, 479].
P. A. Smith (New York, N. Y.).

√✱**Chevalley, C. The Betti numbers of the exceptional simple Lie groups.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 21–24. Amer. Math. Soc., Providence, R. I., 1952.

It has been known for some time that the problem of determining the Betti numbers of compact Lie groups can be reduced to a purely algebraic problem, namely, determining the algebra $H(\mathfrak{g})$ of adjoint-invariant skew symmetric multilinear forms on the Lie algebra $\mathfrak{g}$ of the compact Lie group G. This algebraic problem was first solved for the classical groups by R. Brauer [C. R. Acad. Sci. Paris **201**, 419–421 (1935)] in a case by case treatment. In this paper, the author presents some general methods introduced by himself, Koszul, and A. Weil. As an application, the Betti numbers of the exceptional Lie algebras can be computed.

The major steps are as follows. 1. It follows from a result of Hopf (proved transcendentally by Hopf, algebraically by Koszul) that $H(\mathfrak{g})$ is the exterior algebra over a subspace $P(\mathfrak{g})$ which is spanned by forms of odd degrees $p_1, \cdots, p_l$. These are called "primitive exponents." The problem is thus to determine the primitive exponents. 2. Let $I(\mathfrak{g})$ be the space of adjoint-invariant symmetric polynomials on $\mathfrak{g}$, $\omega_1, \cdots, \omega_n$ linear forms on $\mathfrak{g}$, $P \epsilon I(\mathfrak{g})$. Then the mapping $P \to \eta = (1/n)\sum_{i=1}^{n}(\partial P/\partial x_i)\,(d\omega_1, \cdots, d\omega_n)\omega_i$, first considered by Weil, is a linear mapping of $I(\mathfrak{g})$ onto $P(\mathfrak{g})$ whose kernel is the subspace of $I(\mathfrak{g})$ spanned by 1 and products of homogeneous invariant elements of degrees >0. Let $p_k = 2q_k - 1$ $(k=1, \cdots, l)$. Since $\deg P = 2\deg \eta - 1$, and $I(\mathfrak{g})$ is generated by l algebraically independent elements $I_1, \cdots, I_l$, the problem is reduced to determining the algebraic structure of $I(\mathfrak{g})$. 3. Since the orbit of any maximal toroidal subgroup H of G under inner automorphisms covers G, $I(\mathfrak{g})$ is determined by $I'(\mathfrak{g})$, the ring of polynomial functions on $\mathfrak{H}$, the Lie algebra of H, which are the restrictions of polynomials in $I(\mathfrak{g})$ to $\mathfrak{H}$. Moreover, $I'(\mathfrak{g})$ is identical with the $S_w(\mathfrak{H})$, the set of polynomials on $\mathfrak{H}$ which are invariant under inner automorphisms by elements of the normalizer of H. (This group of automorphisms, the so-called Weyl group, can be determined from the root diagram.)

Thus the problem is reduced to the solution of the "Formenproblem" for the (finite) Weyl group W. Knowing in advance that

$$p_1 + p_2 + \cdots + p_l = \dim G, \quad q_1 \cdot q_2 \cdot \cdots \cdot q_l = \text{order of } W,$$

the author finds by explicit computation that the primitive exponents are for

G_2: 3, 11
F_4: 3, 11, 15, 23
E_6: 3, 9, 11, 15, 17, 23
E_7: 3, 11, 15, 19, 23, 27, 35
E_8: 3, 15, 23, 27, 35, 39, 47, 59.

Combining these results with results for the classical groups, the author remarks that $p_{k+1} - p_k = p_{l-k+1} - p_{l-k}$ $(1 \leqq k < l)$, and announces that explanation of this phenomenon is lacking. *G. D. Mostow* (Syracuse, N. Y.).

Freudenthal, H. La structure des groupes à deux bouts et des groupes triplement transitifs. Nederl. Akad. Wetensch. Proc. Ser A. **54** = Indagationes Math. **13**, 288–294 (1951).

The author gives a proof of the theorem [also proved, unknown to the author, by Iwasawa, Ann. of Math. **54**, 345–348 (1951); these Rev. **13**, 206] that a connected, separable, locally compact group G with two ends is direct product of a compact group and the real line. By a well known theorem of the author it is sufficient to show that G is maximally almost periodic. An element a exists whose positive and negative powers approach the two end-points of G respectively; with respect to the left translations by the powers of this element, G has a fundamental domain K with compact closure. If b, c are any two points of G (which can be assumed to lie in K), a real function on K is constructed which separates b from c; the function is then extended by periodicity, i.e. by $f(ax) = f(x)$, and is then shown to be almost periodic. As an application the author gives an improved version of a theorem of J. Tits [Compositio Math. **9**, 85–96 (1951); these Rev. **12**, 673] on characterizing the projective group of the line as a triply transitive group.
H. Samelson (Ann Arbor, Mich.).

★**Freudenthal, Hans. Oktaven, Ausnahmegruppen und Oktavengeometrie.** Mathematisch Instituut der Rijksuniversiteit te Utrecht, Utrecht, 1951. i+49 pp.

The author presents, in a very clear and self-contained account, a geometric interpretation of the exceptional Lie groups F_4 and E_6. The picture is: a real form of E_6 is represented as the projective group of a non-Desarguian plane over the Cayley numbers (a 16-dimensional manifold, "lines" being 8-dimensional spheres). F_4 is the elliptic subgroup of E_6 which preserves the triality between the three entities: point, line, Band (synonym for incidence). To get the analogue in the familiar real projective plane, represent points by $\pm$ unit vectors in the euclidean linear space E^3, lines by the unit normals to planes through zero, and the Band of point X to the incident line Y by the unit vectors $\pm X\times Y$. Any two entities X, Y of different genus are defined as incident if and only if $X\cdot Y=0$. The analogue of F_4 is the orthogonal group in E^3.

The author's account begins with a proof of Hurwitz's theorem that any finite dimensional real linear space with 1) a positive definite inner product, 2) a multiplication satisfying $|xy|=|x|\cdot|y|$, and 3) a unity, must be either the real, complex, quaternion, or Cayley numbers ("Oktaven").

He then proves that the algebra of derivations of the Cayley numbers C is the compact form of the exceptional Lie algebra G_2, [the Cayley numbers being, as is well known, the exceptional simple alternative algebra over the reals]. In the course of the proof, the author derives the Triality principle in $D_4=SO(8, R)$ in both infinitesimal and global forms by infinitesimal and global arguments respectively. The global form states: Given any rotation A of D_4, there exist, modulo sign, unique rotations B and C such that $Ax\cdot By=C(x\cdot y)$ for all x, $y\varepsilon C$. The correspondences $\pm A\leftrightarrow\pm B$, $\pm A\leftrightarrow\pm C$ are $2-2$ automorphisms [that is, if $\bar{D}/N=D_4$, where $\bar{D}$ is simply connected, then the correspondences are induced by automorphisms of D which do not keep the two-element kernel N invariant]. These automorphisms correspond to the automorphisms $a\leftrightarrow b$, $a\leftrightarrow c$ of the Lie algebra of D_4 where $ax\cdot y+x\cdot by=c(x\cdot y)$ for all x, $y\varepsilon C$.

The author now turns to a description of J, the exceptional simple Jordan algebra over the reals. This is the algebra of dimension 27 whose elements are 3×3 Hermitian matrices with Cayley numbers as coefficients, multiplication being defined by $X\circ Y=\frac{1}{2}(XY+YX)$. He proves that a linear transformation on J is a derivation if and only if it preserves the bilinear and trilinear forms $(X, Y)=\mathrm{Tr}\, X\circ Y$ and $(X, Y, Z)=\mathrm{Tr}\,(X\circ Y)\circ Z=\mathrm{Tr}\, X\circ(Y\circ Z)$. The algebra of derivations of J is the exceptional Lie algebra F_4 [cf. Chevalley and Schafer, Proc. Nat. Acad. Sci. U. S. A. **36**, 137–141 (1950); these Rev. **11**, 577]. The author then proves that every element of J is conjugate to a diagonal matrix by some automorphism. He then deduces that every irreducible idempotent is conjugate to diag (1, 0, 0) and can be characterized as either an idempotent of trace 1 or else as any element X with $\mathrm{Tr}\,(X)=\mathrm{Tr}\,(X^2)=\mathrm{Tr}\,(X^3)=1$.

Let Π denote the set of irreducible idempotents of J and Π^* be the set of real multiples of elements of Π. Upon taking rays of Π^* as points, and subsets $L_x=\{Y|X\circ Y=0, X, Y\varepsilon\Pi^*\}$ as lines, a projective plane, the "Oktaven projective plane" $P(C)$, is obtained. In the appendix, the author proves that the group preserving the point-line incidence relations, i.e., the projective group, for this plane is (a real form of) E_6. Proof consists of characterizing E_6 as the group of linear transformations on J which preserves

$$\det X=\mathrm{Tr}\, X^3-\tfrac{3}{2}\,\mathrm{Tr}\, X\,\mathrm{Tr}\, X^2+\tfrac{1}{2}\,(\mathrm{Tr}\, X)^3.$$

The trilinear form $\det(X, Y, Z)$ is defined by polarization and is invariant under E_6. Three points X, Y, Z are colinear if and only if $\det(X, Y, Z)=0$. An element $X\varepsilon\Pi^*$ if and only if $\det(X, X, Y)=0$ for all Y. Thus E_6 is a subgroup of the projective group of the Oktaven-projective plane, and in fact coincides with it [cf. Chevalley, C. R. Acad. Sci. Paris **232**, 1991–1993 (1951); these Rev. **12**, 802]. The Oktaven plane has unique fourth harmonics but is non-Desarguian.

On the other hand, a principle of triality can be imposed on $P(C)$ as follows. Each element of Π is regarded as representing three entities of distinct genus: point, line, and "band." Two entities X and Y of distinct genus are defined as being in incidence if $X\circ Y=0$, or, which is equivalent for elements of Π, $(X, Y)=0$. Given any two entities X, Y of same genus, there is a unique entity $X\vee Y$ of each of the other genera in incidence with X and Y. $X\vee Y=1-(X-Y)^2/[1-\mathrm{Tr}\,(X\circ Y)]$. Given two incident entities X and Y, there is a unique entity Z incident with each. Here $X\circ Y=0$ and $Z=X\vee Y=(1-X)\circ(1-Y)$. The group of automorphisms of this latter projective geometry, which preserve incidence relations among the three genera is proved isomorphic to the compact form of F_4.

The reader should ignore the last sentence in the first paragraph of 7.10. The erroneous assertion there is corrected in the appendix. Also, throughout 7 the author employs the term "Oktavenprojectiveebene" to refer to the plane with triality postulates, and he ignores completely the projective plane obtained by considering only the two genera, point and line. The reader will avoid confusion by looking at the Appendix before reading 7.10. Errata: p. 13, l. 7. insert "$\alpha_0=0$ and $\cdots$"; p. 22, l. 11. "δ_i" instead of "δ"; p. 30, 6.3.3 1 3. "non-ε" instead of "ε". *G. D. Mostow.*

★**Iwasawa, Kenkichi. Some properties of (L)-groups.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 447–450. Amer. Math. Soc., Providence, R. I., 1952.

A restatement of the principal theorems proved by the author in an earlier paper [Ann. of Math. (2) **50**, 507–558 (1949); these Rev. **10**, 679], together with some theorems of A. Borel and Matsushima. These results not only show the importance of the conjecture that every connected locally compact group is a projective limit of Lie groups but also will probably turn out to be important steps in its final proof. *A. M. Gleason* (Washington, D. C.).

Nakamura, Masahiro, and Takeda, Zirô. Group representation and Banach limit. Tôhoku Math. J. (2) **3**, 132–135 (1951).

The authors join the rapidly growing group of those who have (independently) made use (almost verbatim) of a proof of B. Sz.-Nagy [Acta Univ. Szeged. Sect. Sci. Math. **11**, 152–157 (1947); these Rev. **9**, 191] to prove some variant of: If a group G has an invariant mean on its bounded real functions, then every bounded representation of G over a Hilbert space is equivalent to a unitary representation. The present version and that of Dixmier [ibid. **12A**, 213–227 (1950); these Rev. **12**, 267] also assume a topology on the group, continuity for the real functions, and strong continuity for the given bounded representation. The reviewer's version [Trans. Amer. Math. Soc. **69**, 276–291 (1950); these Rev. **13**, 357] is the one stated above.

M. M. Day (Urbana, Ill.).

Mackey, George W. Induced representations of locally compact groups. I. Ann. of Math. (2) **55**, 101–139 (1952).

In an earlier note [Proc. Nat. Acad. Sci. U. S. A. **35**, 537–545 (1949); these Rev. **11**, 158] the author generalized the notions of imprimitivity and induced representations to unitary representations of locally compact groups G. The present paper is the first of a projected series in which the author plans to investigate in detail these generalizations and the closely related questions of the relations of the unitary representations of G and those of closed subgroups. One of the main results (theorem 12.1) expresses a given induced representation as a (generalized direct) sum over certain double cosets; similarly (theorem 12.2) the Kronecker product of two induced representations. Another main result (theorem 13.1) provides a usable formula for computing the strong intertwining number of two given induced representations. These theorems imply certain generalizations of the classical Frobenius reciprocity theorem; and in the case where all irreducible constituents are discrete and finite-dimensional the author's generalizations seem to be best possible in the sense that they are as strong as the classical results of Frobenius and others. Another generalization has been given by the reviewer [ibid. **37**, 431–435 (1951); these Rev. **13**, 205] in the case where the subgroup is compact; thus there now exist two different kinds of generalization of the Frobenius reciprocity theorem. This suggests the possibility of a further generalization which should contain both the author's and the reviewer's. In the reviewer's opinion this problem involves considerable new difficulties.

The paper is organized as follows. Part I deals with the definition and elementary properties of induced representations; it also contains the solutions of various preliminary difficulties, such as for instance the study of quasi-invariant measures on homogeneous spaces. Part II contains that special case of the author's results where only discrete direct sums occur. Part III completes the proofs of the main results in the general case. In part IV applications are given to semidirect products, to the group of motions in the plane and to the $n\times n$ unimodular group. The latter contain some results of Gelfand and Neumark and also improvements of these. The same methods can be applied to complex semisimple Lie groups in general. Some of the results of this paper have been announced earlier [C. R. Acad. Sci. Paris **230**, 808–809 (1950); these Rev. **11**, 580]. Closely related topics have been discussed by the author in another paper [Amer. J. Math. **73**, 576–592 (1951); these Rev. **13**, 106].

F. I. Mautner (Baltimore, Md.).

Mautner, F. I. Fourier analysis and symmetric spaces. Proc. Nat. Acad. Sci. U. S. A. **37**, 529–533 (1951).

Soient G un groupe localement compact, K un sous-groupe compact de G et supposons l'existence sur G d'une involution $g\to\check{g}$ telle que: 1) $k=\check{k}$ pour $k\epsilon K$; 2) tout g est de la forme ks avec $\check{s}=s^{-1}$. Les fonctions définies sur G/K s'identifient aux fonctions sur G vérifiant $\varphi(kg)=\varphi(g)$. Donc on peut définir une représentation unitaire $g\to U_g$ de G dans $L^2(G/K)$ et il est clair que l'anneau commutant des U_g est engendré par les opérateurs $\varphi\to\varphi*\theta$ où $\theta(kgk')=\theta(g)$. Or une telle fonction θ vérifie (en raison de l'existence de $g\to\check{g}$) $\theta(\check{g})=\theta(g^{-1})$, en sorte que l'algèbre formée par ces fonctions est commutative (ce raisonnement ingénieux est dû à Gelfand et Neumark, qui l'ont utilisé dès 1948 [voir Naĭmark, Uspehi Matem. Nauk (N.S.) **3**, no. 5(27), 52–145 (1948); ces Rev. **10**, 308; voir en particulier p. 134]). Donc le commutant de la représentation considérée est commutatif, d'où l'auteur déduit que: 1) la décomposition "centrale" de cette représentation fait intervenir presque partout des facteurs de type (I); 2) dans presque toute composante irréductible, le sous-espace des vecteurs invariants par K est de dimension 1; 3) on a pour les fonctions de $L^2(G/K)$ une formule de Plancherel du type $(\varphi,\psi)=\int \mathrm{Tr}\,[\phi(t)\psi(t)^*]dm(t)$. L'auteur explicite cette formule en choisissant dans chaque t-composante une base orthonormale dépendant mesurablement de t. En particulier, on obtient pour les fonctions $\varphi(g)$ de carré sommable et vérifiant $\varphi(kgk')=\varphi(g)$ un développement suivant les fonctions "sphériques" de Gelfand. L'auteur obtient ces résultats comme conséquence de son théorème général sur les représentations induites [Proc. Nat. Acad. Sci. U. S. A. **37**, 431–435 (1951); ces Rev. **13**, 205] et du théorème de Plancherel pour les groupes unimodulaires arbitraires. Note du rapporteur: soit **A** l'algèbre commutative avec involution formée des fonctions continues à support compact sur G telles que $f(kxk')=f(x)$; il est trivial de vérifier que le théorème de Plancherel "abstrait" démontré par le rapporteur [Ann. of Math. (2) **53**, 68–124 (1951); ces Rev. **12**, 421] s'applique à la forme positive $f\to\epsilon(f)=f(e)$ sur **A**; donc on a $\epsilon(fgh)=\int_\sigma\theta(fgh)d\theta$ où σ est un ensemble localement compact de "caractères" $f\to\theta(f)$ de **A** et où $d\theta$ est une mesure positive sur σ (on note fg le produit de composition). Si l'on désigne par $f\to U_f$ la représentation unitaire de **A** définie par ϵ, U_f est l'operateur $g\to fg$ sur le sous-espace des $g\epsilon L^2(G)$ telles que $g(kxk')=g(x)$, de sorte que si $f\epsilon\mathbf{A}$ est de type positif on a $U_f\geqq 0$; comme les $\theta\epsilon\sigma$ proviennent des "caractères" de l'algèbre uniformément fermée engendrée par les U_f on en conclut que $\theta(f)\geqq 0$ dès que $f\epsilon\mathbf{A}$ est de type positif; donc pour toute g continue à support compact sur G on a $\theta[(\tilde{g}g)^0]\geqq 0$ où l'on pose en général $g^0(x)=\int\int g(kxk')dkdk'$; on déduit immédiatement de là que pour tout $\theta\epsilon\sigma$ il existe une (et une seule) fonction de type positif $\theta(x)$ sur G vérifiant $\theta(f)=\int f(x)\overline{\theta(x)}dx$ pour $f\epsilon\mathbf{A}$ et $\theta(kxk')=\theta(x)$; puisque $\theta(fg)=\theta(f)\theta(g)$ dans **A**, on voit que toutes les $\theta(x)$ sont des fonctions sphériques élémentaires [Gel'fand, Doklady Akad. Nauk SSSR (N.S.) **70**, 5–8 (1950); ces Rev. **11**, 498]. Maintenant, appliquons la formule $\epsilon(fgh)=\int\theta(fgh)d\theta$ en y remplaçant h par h^0, où h est une fonction continue à support compact variant de telle sorte que la mesure $h(x)dx$ converge vers ϵ (masse $+1$ en e); on trouve à la limite $\epsilon(fg)=\int\theta(fg)d\theta$ pour f, $g\epsilon\mathbf{A}$; remplaçant g par $\tilde{g}$ cela s'écrit $\int f(x)g(x)dx=\int\hat{f}(\theta)\overline{\hat{g}(\theta)}d\theta$ où l'on pose d'une façon générale $\hat{f}(\theta)=\theta(f)=\int f(x)\overline{\theta(x)}dx$. Pour obtenir une formule applicable aux fonctions de $L^2(G/K)$ on remarque tout d'abord que, comme dans le cas abélien, on a une "formule d'inversion de Fourier": si $f\epsilon\mathbf{A}$ est de type positif, alors $\int\hat{f}(\theta)d\theta<+\infty$ et $f(x)=\int\theta(x)\hat{f}(\theta)d\theta$ pour tout $x\epsilon G$. Ceci fait, prenons g continue à support compact vérifiant $g(kx)=g(x)$; alors $g\tilde{g}$ est dans **A** et de type positif; donc d'après ce qui précède on a $\int|g(x)|^2dx=\epsilon(g\tilde{g})=\int\theta(g\tilde{g})d\theta$. Mais soit $\{H(\theta), U_x(\theta)\}$ la représentation unitaire irréductible de G définie par $\theta(x)$ et posons $U_g(\theta)=\int U_x(\theta)g(x)dx$; puisque $g(kx)=g(x)$ il est clair que $U_g(\theta)$ applique $H(\theta)$ dans le sous-espace des vecteurs invariants par les $U_k(\theta)$, lequel est de dimension un puisque $\theta(x)$ est élémentaire (voir la note de Gelfand); de là résulte par un calcul trivial que $\theta(g\tilde{g})=\mathrm{Tr}\,[U_g(\theta)U_g(\theta)^*]$ (trace usuelle), et tous les résultats de l'auteur sont démontrés. En outre, notre méthode (trouvée en 1950): 1) introduit un "objet dual" σ qui généralise directement le groupe dual d'un groupe abélien; 2) évite entièrement les nombreux ensembles "exceptionnels" de mesure nulle que l'auteur ne peut éviter; 3) ne

suppose rien connu de la théorie générale des "direct integrals" ni de celle des représentations induites; 4) ne repose sur aucune hypothèse de séparabilité. On pourrait aussi du reste utiliser le théorème de Plancherel pour les systèmes de translations généralisées [Lewitan, C. R. (Doklady) Acad. Sci. URSS (N.S.) **47**, 318–321 (1945); ces Rev. **7**, 127] ou encore les méthodes bien connues applicables aux groupes abéliens, qui s'adaptent presque trivialement à la situation présente. *R. Godement* (Nancy).

✶Eberlein, W. F. Spectral theory and harmonic analysis. Proceedings of the Symposium on Spectral Theory and Differential Problems, pp. 209–219. Oklahoma Agricultural and Mechanical College, Stillwater, Okla., 1951. $3.00.

L'auteur rappelle les faits fondamentaux de la théorie spectrale dans les groupes abéliens localement compacts: identification du groupe dual à l'ensemble des homomorphismes de L^1 sur le corps complexe, théorème taubérien de Wiener, théorème de Beurling, synthèse spectrale quand la frontière du spectre est dénombrable, ou quand les fonctions considérées sont soit presque-périodiques (au sens classique) soit de type positif. L'article se termine par quelques considérations sur les fonctions faiblement presque-périodiques, et par une bibliographie complète. *R. Godement.*

Stoilov, S. On the factorization of locally Euclidean topological groups by their closed zero-dimensional subgroups. Acad. Repub. Pop. Române. Bul. Şti. A. **1**, 829–834 (1949). (Romanian. Russian and French summaries)

Let G be a locally euclidean topological group and P a zero-dimensional subgroup. The author makes certain observations concerning the relation between the spaces G and G/P. For example, if T is the canonical mapping $G\rightarrow G/P$ and σ is a simple arc in G/P, the components of $T^{-1}\sigma$ are simple arcs. This is true whether or not P is discrete. *P. A. Smith* (New York, N. Y.).

Vilenkin, N. Ya. Direct and inverse spectra of topological groups and their character theories. Izvestiya Akad. Nauk SSSR. Ser. Mat. **15**, 503–532 (1951). (Russian)

This is in large part an exposition of results announced earlier [Doklady Akad. Nauk SSSR (N.S.) **72**, 617–620 (1950); these Rev. **12**, 79] using results of a recent paper on character groups [same Izvestiya **15**, 439–462 (1951); these Rev. **13**, 319]. There follows an indication of results not covered in the reviews above, using terms which are defined there.

In a first section on inverse spectra it is proved that if all the groups of the spectrum are involutory then the limit group is also involutory. The second section on direct spectra analyzes the relation of limit groups of two spectra, one of which may be regarded as a subspectrum of the other. To this section belongs the theorem (5): Let $(G_a;\varphi_{ab})$ be an inverse spectrum. If all the maps φ_{ab} are bounded, if the fixed group G_e is quasiconvex, and if all the maps φ_{eb}, $b>e$, e fixed, are complete isomorphisms, then the limit group of the spectrum is completely isomorphic to the fixed group G_e.

A direct spectrum is called closed if the map of G_b, for every b, into the limit group F is a complete isomorphism onto a closed subgroup of F. It is proved (theorem 7) that if a quasi-convex group G is approximated (in a certain sense) by a subset of its closed subgroups, then these subgroups generate in a natural way a direct closed spectrum whose limit group is completely isomorphic to the given group G. Section 3 analyzes the dual relation of the limit groups of a direct spectrum of groups with bounded structure and the indirect spectrum of their dual groups. In a fourth section there is a discussion of the related direct spectra defined by Čogošvili, and by Freudenthal. It is shown that there is a continuous algebraic isomorphism of the limit groups here defined into those of Čogošvili, and that two spectra give rise to isomorphic limit groups in one case if and only if they do so also in the other.

L. Zippin (Brooklyn, N. Y.).

Vilenkin, N. Ya. Direct operations on topological groups. Mat. Sbornik N.S. **29**(71), 371–402 (1951). (Russian)

Some of the material of this paper was announced in a preliminary note [Doklady Akad. Nauk SSSR (N.S.) **65**, 3–5 (1949); these Rev. **10**, 507] and is also discussed in a general expository paper [Uspehi Matem. Nauk (N.S.) **5**, no. 4(38), 19–74 (1950); these Rev. **12**, 78].

Given an infinite class of topological groups and two distinct types of infinite direct products of topological groups, it is possible to construct classes of groups analogous to the Borel and the Souslin classes of topological sets. The first appearance of this idea is credited to Köthe and Toeplitz (for groups all isomorphic to the additive reals) [J. Reine Angew. Math. **171**, 193–226 (1934)]. For the pair of "direct operations" corresponding to the forming of the direct topological product and the direct algebraic product (see the first cited review) it is shown that there exist groups of arbitrarily large Borel class belonging to no previous Borel class, and "Souslin" groups which are not "Borelian". The process of forming any one of these classes is called a direct operation; applied to a given system of groups it yields a "limit" group. The processes here defined are shown to terminate in the "Souslin-operation".

The theory is connected to that of the character groups of abelian groups. It is shown that to every direct operation (on a given system of abelian groups) there may be constructed a "complementary" direct operation which (applied to the groups dual to those of the given system) yields the dual of the limit group of the first operation. The author shows that the class of fibered groups, introduced by him [Mat. Sbornik N.S. **24**(66), 189–226 (1949); these Rev. **10**, 679], is the largest class of groups to which the Pontryagin duality theory carries over in full in this way.

L. Zippin (Flushing, N. Y.).

Vilenkin, N. Ya. Direct decompositions of topological groups. III. Mat. Sbornik N.S. **29**(71), 519–528 (1951). (Russian)

A group is called of type LD (limit of inverse spectrum of discrete groups) if it has a complete set of neighborhoods of e consisting of invariant subgroups and if it is complete in the sense of Raikov or, equivalently in this case, the sense of Weil. The groups actually studied in this note are abelian. Theorem 1: For a topological abelian group G to be a topological direct sum of a countable set of infinite cyclic groups it is necessary and sufficient that 1) G is LD, 2) every discrete factor group of G has a finite set of generators, 3) every element of G generates an infinite discrete subgroup, 4) G satisfies the 2nd countability axiom. Theorem 2: For G to be the direct sum of a countable set of cyclic groups, whether finite or infinite, it is necessary and sufficient that it have properties 1), 2), and 4) above, and also 3') in the subgroup H of elements which generate bicompact subgroups of G there is an everywhere dense set of elements of finite order.

A discrete group is said to have rank n (in the sense of Prüfer) if every finite set of elements of G lies in a subgroup generated by no more than n elements. Theorem 3: For G to decompose into a direct sum of a countable set of additive groups of rational numbers (each taken in the discrete topology) it is necessary and sufficient that 1) G is a limit group of an inverse spectrum of infinitely divisible discrete abelian groups of finite rank, and 2) for every sequence $x_1, x_2, \cdots$, of elements of G and sequence of numbers $m_1, m_2, \cdots$, the relation $\lim m_n x_n = 0$ implies $\lim x_n = 0$. A final theorem combines these with earlier results on decompositions into sums of groups of type p^∞. *L. Zippin.*

✓***Ritt, J. F. Differential groups.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 1, pp. 207–208. Amer. Math. Soc., Providence, R. I., 1952.

A brief indication of the contents of the late author's four papers on the subject of differential groups [Ann. of Math. (2) **51**, 756–765 (1950); **52**, 708–726 (1950); **53**, 491–519 (1951); **54**, 110–146 (1951); these Rev. **11**, 639; **12**, 241, 674; **13**, 207]. *E. R. Kolchin* (New York, N. Y.).

Fuchs, L. The extension of partially ordered groups. Acta Math. Acad. Sci. Hungar. **1**, 118–124 (1950). (English. Russian summary)

In this paper the author generalizes to the case of ordered groups the known method of Schreier for constructing all extensions of one group by another. Conditions under which the ordered group extension is a lattice or totally ordered are also mentioned. *L. Nachbin* (Rio de Janeiro).

Isiwata, Takesi. Non-discrete linearly ordered groups. Kôdai Math. Sem. Rep. **1950**, 84–88 (1950).

This paper considers linearly ordered groups endowed with the natural order topology. The author's aim is to discuss the relationship between the following properties of an ordered group: being locally archimedian, connected, locally compact, and zero- and one-dimensional. *L. Nachbin* (Rio de Janeiro).

Ohnishi, Masao. On linearization of ordered groups. Osaka Math. J. **2**, 161–164 (1950).

In continuation of recent work of L. Fuchs [Amer. J. Math. **72**, 191–194 (1950); these Rev. **11**, 323] and C. J. Everett [ibid. **72**, 216 (1950); these Rev. **11**, 324] the author drops the assumption of the groups being commutative and proves the following results. If G is a group and $a \epsilon G$, call C_a the set of all $x_1 a x_1^{-1} \cdots x_n a x_n^{-1}$, that is the least set containing a closed under products and invariant by inner automorphisms. Assume that (1) C_a and C_a^{-1} do not intersect for any a; (2) if C_a and C_b each intersect C_c, then C_a and C_b do intersect (intersect = have a point in common other than 1). Then the maximal admissible order relations on G are total. Conversely, if G has an admissible total order relation then (1) holds; and if every maximal admissible order relation on G is total then (2) holds. *L. Nachbin.*

NUMBER THEORY

✓***Thébault, Victor. Les récréations mathématiques (parmi les nombres curieux). Avec des notes de A. Buquet.** Gauthier-Villars, Paris, 1952. vi+297 pp. 2500 francs.

This is a reprinting of the author's "Supplément à Mathesis, **54** (1943)" [these Rev. **8**, 134] followed by an equal quantity of new recreational arithmetic in the same style. There are various problems on numbers involving the ten digits of the denary scale once each; e.g., the author exhibits a set of ten such numbers whose sum is 9876543120. In connection with "cyclic" numbers he remarks that the result of transposing the first two digits of 109890 to the end is to multiply this number by 9. He gives a useful list of all the perfect squares whose digits are all different. There are many problems on other scales of notation; e.g., he proves that the binary, quaternary, denary and duodenary are the only scales in which a number consisting of two odd digits cannot be a perfect square. He shows that in any scale there is at least one four-digit square of the form

$$mnpq = (aa)^2, \quad m+p = n+q = a.$$

He finds perfect cubes of the form $abab$ in the scales of 7, 38, 41, 239, 682, $\cdots$; e.g., $13^3 = 2626$ in the septenary scale. In connection with Pythagorean numbers, he remarks that X^2 is the sum of two positive squares if and only if X is of the form

$$x+y+\sqrt{2xy},$$

where x and y are positive integers. He gives general expressions for the sides of a "quasi-rectangular" triangle which has two angles differing by a right angle. (The sides being integers, the circum-diameter and area are likewise integers.) He remarks that the expressions

$$x = 7a^4 - 11ab^3, \quad y = 7b^4 - 2a^3b, \quad z = 7a^4 - 2ab^3, \quad t = 7b^4 - 11a^3b$$

provide an infinity of solutions for the diophantine equation

$$x^3 + y^3 = z^3 + t^3.$$

He closes his own part of this unusual book with a set of one hundred miscellaneous problems and solutions. An appendix by A. Buquet contains two notes: one on the Pell-Fermat equation and one on "arithmotriangulation". *H. S. M. Coxeter* (Toronto, Ont.).

Miller, J. C. P. Large primes. Eureka **1951**, no. 14, 10–11 (1951).

Since 1876 when Lucas announced that $2^{127}-1$ is a prime, many attempts have been made to find a larger prime. This note records the first successful results. The primes identified by the EDSAC are

$$N_k = 1 + k(2^{127} - 1),$$

for k = 114, 124, 388, 408, 498, 696, 738, 774, 780, 934, 978, and finally

$$1 + 180(2^{127} - 1)^2,$$

a number of 79 decimal digits. The time required by the EDSAC for testing this latter number was 27 minutes. The announcement includes also the fact that A. Ferrier has identified the prime $(2^{148}+1)/17$, a number of 44 digits, by use of a desk calculator. All primes mentioned were established by methods based on the converse of Fermat's theorem. *D. H. Lehmer* (Los Angeles, Calif.).

Droop, Goswin. On the classification of Pythagorean triples. Revista Soc. Cubana Ci. Fís. Mat. **2**, 163–169 (1950). (Spanish)

Bini, Umberto. Due aspetti del quoto nel teorema di Fermat. Archimede 3, 189–191 (1951).

The author gives a proof of Fermat's theorem somewhat like Euler's first proof. Both proofs involve the divisibility properties of binomial coefficients. The present proof uses the expansion of x^n+y^n as a polynomial in $x+y$ and xy rather than the binomial theorem. *D. H. Lehmer.*

Verkaart, H. G. A. Deriving a general formula for finding positive and negative integers which satisfy the relation $(-a+b+c)(a-b+c)(a+b-c)=abc$, where a, b and c represent three distinct numbers different from zero. Nieuw Tijdschr. Wiskunde 39 (1951/52), 153–157 (1952). (Dutch)

Skolem, Th. A simple proof for the solvability condition for the Diophantine equation $ax^2+by^2+cz^2=0$. Norsk Mat. Tidsskr. 33, 105–112 (1951). (Norwegian)

A simplified proof for the congruence conditions for the solvability of the equation given in the title by means of Thue's congruence principle; the analogous conditions for solvability in polynomials is also derived. *O. Ore.*

Furquim de Almeida, F. The law of quadratic reciprocity. Bol. Soc. Mat. São Paulo 3, no. 1–2 (1948), 3–8 (1951). (Portuguese)

The author gives a proof of Legendre's law of reciprocity based on the theory of finite fields. The odd primes p and q being given, one considers the finite field of order p^n where $p^n\equiv1 \pmod q$. In this field, the Vandermonde determinant D of order q based on the roots of $x^q=1$ satisfies

$$D^2=(-1)^{(q-1)/2}q^q \quad\text{and also}\quad D^p=\left(\frac{p}{q}\right)D.$$

For D to belong to the field of residues modulo p it is thus necessary and sufficient that on the one hand $(p/q)=+1$ and on the other hand that $(-1)^{(q-1)/2}q^q$ be a quadratic residue of p. This statement reduces simply to

$$(-1)^{(q-1)/2\cdot(p-1)/2}\left(\frac{q}{p}\right)=\left(\frac{p}{q}\right).$$

D. H. Lehmer (Los Angeles, Calif.).

Eljoseph, Nathan. Extensions of Wolstenholme's theorem. Riveon Lematematika 4, 9–15 (1950). (Hebrew. English summary)

Let $S_k(m)$ denote the sum $\sum_t t^{-k}$, where t ranges over the numbers $\leqq m$ and prime to m. The statement that $S_1(p)$ is divisible by p^2 where p is a prime is known as Wolstenholme's theorem. Several generalizations are known. The author proves a few theorems for the case of composite m. For example: If k is odd then $S_k(m)$ is divisible by m^2/R where R denotes the product of primes p dividing m for which $p-1$ divides k and for which p does not divide k or $q-1$ for any prime q dividing m. *D. H. Lehmer* (Los Angeles, Calif.).

Eljoseph, Nathan. Remarks on my paper "Extensions of Wolstenholme's theorem." Riveon Lematematika 4, 59–61 (1950). (Hebrew. English summary)

A simple proof, not using the properties of Bernoulli numbers, is given of the fact that the sum of the kth powers of the numbers less than and prime to p^s is congruent modulo p^s either to $\phi(p^s)$ or to 0 according as $p-1$ does or does not divide k. *D. H. Lehmer* (Berkeley, Calif.).

Zakay, Shlomo, and Jarden, Dov. Simple proof of Leudesdorf's theorem in cases of a modul non-divisible by 6. Riveon Lematematika 4, 16–17 (1950). (Hebrew. English summary)

Let S denote the sum of the reciprocals of those integers $\leqq m$ which are prime to m. The authors point out that the results

(a) S is divisible by m^2 if m is prime to 6,
(b) S is divisible by $m^2/3$ if $m=3(2n+1)$,
(c) S is divisible by $m^2/4$ if $m=2^\alpha$,

follow easily from certain manipulations with the terms of S. The result

(d) S is divisible by $m^2/2$ if m is even but not divisible by 3

requires some further reasoning. The final case of m divisible by 6 is not discussed. [See Hardy and Wright, An introduction to the theory of numbers, Oxford, 1938, chap. 8.]

D. H. Lehmer (Los Angeles, Calif.).

Szász, G. Über die Teilersumme der Zahlen von der Form $2^\alpha p^\beta$. Elemente der Math. 6, 135–136 (1951).

The author notes that the number $N=2^\alpha p^\beta$, where p is an odd prime, is abundant or deficient according as $2^\alpha p$ is abundant or deficient. Also, N is deficient for $p>2^{\alpha+1}$ and perfect or abundant for $p<2^{\alpha+1}$. Perfection occurs only when $\beta=1$ and $p=2^{\alpha+1}-1$. *D. H. Lehmer.*

Erdös, P. On integers of the form 2^k+p and some related problems. Summa Brasil. Math. 2, 113–123 (1950).

Let $c_1, c_2, \cdots$ denote positive absolute constants. Let $f(n)=\sum_{2^k+p=n}1$. Thus $c_1x\leqq\sum_{n\leqq x}f(n)\leqq c_2x$ for $x>2$. Romanoff proved (1) $\limsup x^{-1}\sum_1^x f^k(n)<\infty$ for $k=2$ and hence (2) $\sum_{n\leqq x,\, f(n)>0}1\geqq c_3x$ [cf. Landau, Über einige neuere Fortschritte der additiven Zahlentheorie, Cambridge Univ. Press, 1937]. The author proves (1) for every k. He proves the existence of an infinite number of n's such that $f(n)>c_4\ln\ln n$ [thus $\limsup f(n)=\infty$], and constructs on the other hand an arithmetic progression of odd numbers no term of which has the form 2^k+p. Finally, generalizing (2), he proves: Given an infinite sequence $a_1<a_2<\cdots$ of integers such that $a_k\mid a_{k+1}$ [$k=1, 2, \cdots$], then $\liminf x^{-1}\sum_{n\leqq x,\, n=p+a_k}1>0$ if and only if both (3) $\limsup \ln a_k/a_k<\infty$ and (4) $\sum_{d\mid a_k}1/d<c_5$.

Formula (1) is derived from (5) $\sum_1^\infty B^{v(d)}/dl_2(d)<\infty$. Here $B>0$ is constant; $v(d)=\sum_{p\mid d}1$; $l_2(d)$ is the exponent of 2 (mod d). The proof of (5) follows the pattern of the case $B=1$ [cf. Landau, loc. cit., p. 68]. The sufficiency proof of the last theorem is based on the following lemma: (3) and (4) imply $\sum_{l<k}\sum_{d\mid(a_k-a_l),\,(d,a_k)=1}1/d<c_6k$. *P. Scherk.*

Borel, Émile. Sur une propriété arithmétique des suites illimitées d'entiers. C. R. Acad. Sci. Paris 233, 769–770 (1951).

Let S be an infinite sequence of integers. The author denotes by N_k the number of distinct integers of k digits obtained by taking the last k digits of the members of S. Thus if S is the sequence of odd primes $N_1=4$, $N_2=40$, $N_3=400$, etc. A sequence S is called quasi-normal in case $N_{k+1}=10N_k$ for all $k\geqq K_0$. By Dirichlet's theorem, the sequence of primes is quasi-normal. The sequence of squares, however, is not, nor is the sequence of any powers. The author raises the question about the sequence consisting of the numerators of the successive convergents of the continued fraction for $\sqrt{2}$. He notes that for this sequence $N_1=4$, $N_2=22$, $N_3=109$. *D. H. Lehmer.*

Niven, Ivan, and Zuckerman, H. S. On the definition of normal numbers. Pacific J. Math. 1, 103–109 (1951).

Let R be a real number with fractional part $0.\,x_1, x_2, \cdots$ when written to the scale r. Let $N(b,n)$ denote the number of occurrences of the digit b in the first n places. R is said to be simply normal if for each b, $0 \leqq b < r$,

$$\lim_{n\to\infty} N(b, n)/n = 1/r.$$

Borel defined $\mathcal{R}$ to be normal in the scale r if all the numbers $\mathcal{R}, r\mathcal{R}, r^2\mathcal{R}, \cdots$ are simply normal to all the scales $r, r^2, r^3, \cdots$. Borel also stated that a characteristic property of a normal number is the following: Denote by B a sequence of v digits b_i, $0 \leqq b_i < r$, $i = 1, 2, \cdots, v$. Let $N(B, n)$ denote the number of occurrences of the sequence B in the first n decimal places. Then for any B

$$\text{(1)} \qquad \lim_{n\to\infty} N(B, n)/n = 1/r^v.$$

Hardy and Wright [An introduction to the theory of numbers, Oxford, 1938] state without proof that (1) is equivalent to the definition of normality. It is indeed easy to see that a normal number has property (1), but the converse is by no means obvious. The authors prove that (1) is indeed equivalent to normality. *P. Erdös* (Aberdeen).

Niven, Ivan. The asymptotic density of sequences. Bull. Amer. Math. Soc. 57, 420–434 (1951).

The author gives a comprehensive survey of the recent work on the asymptotic density of sequences and also a short discussion of some of the results on Schnirelmann density. Several new results are proved. Denote by $d_2(A)$ the upper density of the sequence A; let A_p denote the subset of integers n of A for which $p|n$, $p^2\nmid n$. Then if $\sum_{i=1}^{\infty} 1/p_i = \infty$, $d_2(A) \leqq \sum_{i=1}^{\infty} d_2(A_{p_i})$. The result is false for the lower density $d_1(A)$. Some applications of this result are given. The author also proves that the density of the multinomial coefficients $n!/a_1! \cdots a_r!$, where $n = \sum_{i=1}^{r} a_i$, $a_i < n-1$ have density 0. *P. Erdös* (Aberdeen).

Mann, Henry B. On the number of integers in the sum of two sets of positive integers. Pacific J. Math. 1, 249–253 (1951).

Let $A, B, \cdots$ be sets of non-negative integers. Let $A^0, B^0, \cdots$ denote the union of $A, B, \cdots$ and the number 0, let $A(n)$ denote the number of integers in A not exceeding n, and let $A+B$ be the set $\{a+b \,|\, a\epsilon A, b\epsilon B\}$. Put

$$\alpha = \inf A(n)/n, \qquad \alpha^* = \inf A(n)/(n+1),$$
$$\bar\alpha = \liminf A(n)/n, \qquad \alpha_1 = \inf_{n \geqq k} A(n)/(n+1),$$

where k is the smallest positive integer not in A. The author proved in a previous paper [Ann. of Math. (2) 43, 523–527 (1942); these Rev. 4, 35] the following theorem: Put $C = A^0 + B$ if 1 is in B, $C = A^0 + B^0$ if 1 is not in B. If n is not in C we have (1) $C(n) \geqq \alpha^* n + B(n)$. Here α^* can not be replaced by α. The author now shows (2) $C(n) \geqq \alpha_1 n + B(n)$. Since $\alpha_1 \geqq \alpha^*$, (2) is stronger than (1). The author discusses some consequences of (2). *P. Erdös* (Aberdeen).

★**Selberg, Atle. The general sieve-method and its place in prime number theory.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 1, pp. 286–292. Amer. Math. Soc., Providence, R. I., 1952.

This is a clear exposition of the general sieve-method due to the author. The method includes Brun's method and the improvements of it as a special case. The formulation of the technique is not only considerably simpler than in the older methods, but it also leads to better results.

Let there be given a finite set of integers n_i and a set of r primes p_i. Let $N_r = N(p_1, p_2, \cdots, p_r)$ denote the number of n's not divisible by any p_i. Then $N_r = \sum_n \sum_{d|n} \mu(d)$, where $\mu(d)$ is the Möbius-function, and each d is composed only of primes p_i. The problem of the sieve-method is to find upper and lower bounds for N_r. To accomplish this the author introduces ingenious devices for replacing the expression $\sum_{d|n} \mu(d)$ with a similarly built expression which respectively majorizes or minorizes it, and at the same time diminishes the remainder term to a reasonable size. The analysis is reduced to the solution of two extremal problems which, in turn, may be employed to deduce information about the limitations of the method. There are several alternative procedures some of which require extensive numerical calculations.

The older sieve-methods are characterized by their broad generality which makes them yield results where the powerful analytic tools will not work. They lead, however, only to partial and incomplete results. A surprising feature of the present method is that in some special cases the results are the best possible. On the other hand, it is shown that several problems which have been attacked repeatedly by the sieve-method cannot be solved in this way. The author concludes that ". . . it seems that the sieve-method will be of little value for the further progress of these problems in prime number theory which it was originally designed to deal with. But it remains as an extremely general and versatile tool for establishing, for instance, upper bounds, and may perhaps, when in some way combined with an analytic approach, still play an important part in the future of these problems."

A. L. Whiteman (Los Angeles, Calif.).

Haselgrove, C. B. Some theorems in the analytic theory of numbers. J. London Math. Soc. 26, 273–277 (1951).

The author states four main results and eight others required for their proof; he hopes to publish the proofs in due course. His first main result is that if $63/64 < \vartheta < 1$ then there exists n_0 such that each odd $n > n_0$ is representable as the sum of three primes each of which differs from $n/3$ by no more than n^ϑ. This result is proved by an extension of Linnik's method for proving the Goldbach-Vinogradov three prime theorem and consequently depends on obtaining suitable bounds for the number of zeros of $L(s, \chi)$ in rectangles of the form $\beta < \sigma < 1$, $|t - T| < U$. Such an estimate is obtained from an upper bound for $\int_{T-U}^{T+U} |L(\frac{1}{2}+it, \chi)|^4 dt$ by following a paper of Ingham. Finally, this integral is estimated by using a method of Carlson and a new estimate for the sum $\sum_{n<\xi} d(n)d(n+k)\chi(n)\bar\chi(n+k)$.

The second main result generalizes Hoheisel's result that there is a positive $\varphi < 1$ such that for each large n there is a prime p such that $|p-n| < |n|^\varphi$; the author's result deals with Gaussian primes p and complex numbers n and he remarks that this result can be extended to all algebraic number fields. The proof of this result depends on properties of a zeta function studied by Hecke. Zero-free regions for this function similar to those known for $\zeta(s)$ are obtained as well as bounds for the number of zeros in rectangles of the kind previously mentioned. *L. Schoenfeld.*

Erdös, P. On the sum $\sum_{k=1}^{x} d(f(k))$. J. London Math. Soc. 27, 7–15 (1952).

Let $d(n)$ denote the number of divisors of a positive integer n, and let $f(k)$ be an irreducible polynomial with integer coefficients. The author proves that there exist two positive

constants c_1 and c_2 such that $c_1 x \log x < \sum_{k \leqq x} d(f(k)) < c_2 x \log x$ for $x \geqq 2$. For $f(k)$ a quadratic the result, and even an asymptotic equality, had previously been established by the reviewer and Shapiro [Ann. of Math. (2) **49**, 333–340 (1948); these Rev. **9**, 414]. The lower inequality is readily established, but the upper inequality requires some extremely ingenious and involved estimations. In the course of the proof, an earlier result of van der Corput, $\sum_{k \leqq x} d(f(k)) = O(x \log x^c)$, where c depends upon $f(k)$, is used. It is further stated that Brun's method combined with the technique of the paper would enable one to establish $\sum_{p \leqq x} d(f(p)) = O(x)$, where p runs through the primes p less than or equal to x.

R. Bellman (Stanford University, Calif.).

Chowla, S., and Erdös, P. A theorem on the distribution of the values of L-functions. J. Indian Math. Soc. (N.S.) **15**, 11–18 (1951).

Let (d/n) be the Kronecker symbol and

$$L_d(s) = \sum_{n=1}^{\infty} n^{-s}(d/n)$$

for real $s>0$. For fixed $s>0$, let $g(a, x)$ be the number of integers d such that $1 \leqq d \leqq x$, d is congruent to 0 or 1 (mod 4), d is not a square and $L_d(s) < a$. The authors prove that if s is a fixed number greater than $\frac{3}{4}$, then $\lim_{x\to\infty} 2g(a, x)/x = g(a)$ exists; further, $g(0)=0$, $g(\infty)=1$ and $g(a)$ is continuous and strictly increasing. One consequence of this result is that $L_d(s)>0$ for "almost all" d. The proof depends on Polya's estimate for character sums as well as special properties of the sub-series $\sum n^{-s}(d/n)$ in which the summation is extended over all positive integers n whose greatest prime factor does not exceed a given number t. *L. Schoenfeld.*

Gel'fond, A. O. On integral valuedness of analytic functions. Doklady Akad. Nauk SSSR (N.S.) **81**, 341–344 (1951). (Russian)

Let E, $E_1 = \{\alpha_1, \alpha_2, \cdots\}$, and $E_2 = \{\beta_1, \beta_2, \cdots\}$ be three enumerable sets of complex numbers with ∞ as only limit point and such that E consists of all sums $\alpha_i + \beta_j$. Write

$$N_1(r) = \sum_{|\alpha_i| \leqq r} 1, \quad N_2(r) = \sum_{|\beta_j| \leqq r} 1, \quad N(r) = \min(N_1(r), N_2(r)).$$

Denote by $f(z)$ an integral function, by $M(r)$ the maximum of $|f(z)|$ on $|z|=r$, and by K an algebraic field of finite degree ν. Assume that $f(\gamma)$, for all $\gamma \varepsilon E$, is an integer in K with the following property: There exists to every $\delta>0$ a $C_0(\delta)>0$ independent of γ such that $f(\gamma)$ and all its conjugates with respect to K are $< C_0 M(|\gamma|)^{1+\delta}$ in absolute value. The author proves that then two constants θ and λ can be given such that, if

$$\log M(\theta r) < \lambda N(r),$$

then $f(z)$ satisfies a functional equation

$$\sum_{k=1}^{m} A_k f(2+\beta_k) = 0, \qquad m>1,$$

where $A_1, \cdots, A_m$ are non-zero rational integers. (It suffices to take $\theta>4$, $\lambda > 4(\nu+1)\log(\frac{1}{2}\theta - 1)$.) *K. Mahler.*

Schoeneberg, Bruno. Über die Weierstrass-Punkte in den Körpern der elliptischen Modulfunktionen. Abh. Math. Sem. Univ. Hamburg **17**, 104–111 (1951).

Let $\mathfrak{F}$ be an algebraic function field of one variable of genus $p(\mathfrak{F}) = p > 0$. Let $\mathfrak{p}$ be a point of the associated Riemann surface. Moreover, let $\mathfrak{G}$ denote the group of automorphisms of $\mathfrak{F}$, and let $\mathfrak{p}$ be a fixed point for the automorphism S of order n. Let $\mathfrak{C}$ be the subfield of $\mathfrak{F}$ which is invariant for the automorphisms S^ν, $\nu = 1, 2, \cdots, n$, and let the genus of $\mathfrak{C}$ be $p(\mathfrak{C})$. The author proves that $\mathfrak{p}$ is a Weierstrass point if the condition

$$0 \leqq p(\mathfrak{F}) - np(\mathfrak{C}) < n$$

is not satisfied.

This criterion is used to obtain the following results for $\Gamma(N)$, the principal congruence subgroups of Stufe N of the full modular group $\Gamma(1)$. 1) For $N \geqq 7$ the rational points of the fundamental region of $\Gamma(N)$ are Weierstrass points of the algebraic function field belonging to $\Gamma(N)$ [see also Petersson, Arch. Math. **2**, 246–250 (1950); these Rev. **12**, 394]. 2) The points equivalent to i under $\Gamma(1)$ are Weierstrass points for $N=9$ and $N>10$. For $N \leqq 8$, they are not. 3) The points equivalent to $\rho = e^{(2\pi i)/3}$ under $\Gamma(1)$ are Weierstrass points for $N=10$, 12 and $N>13$. For $N \leqq 8$ they are not. 4) ρ and equivalent points are Weierstrass points for $N=13$. *W. H. Simons* (Vancouver, B. C.).

Weil, André. Sur la théorie du corps de classes. J. Math. Soc. Japan **3**, 1–35 (1951).

The paper begins with a very compact formulation of the class field theory which is essential for putting the author's problem into its proper perspective. Let k be an algebraic number field, and denote by A_k the maximal abelian extension of k. The Galois group G_k' of A_k/k is topologized by the Krull topology in which it is a compact totally disconnected topological group. Let C_k denote the idèle class group of k. The class field theory establishes a canonical homomorphism of C_k onto G_k' which is continuous and open when C_k is endowed with the topology derived in a natural fashion from the valuation metrics on the local unit groups. The kernel D_k of this canonical homomorphism is the connected component of the identity in C_k. Thus, if $C_k' = C_k/D_k$, the class field theory establishes a canonical isomorphism between the topological groups C_k' and G_k' by means of which these groups are frequently identified in the sequel.

If K/k is a finite normal algebraic extension, A_K contains A_k and is normal over k with the Galois group $G'_{K,k}$. The Galois group G_K' of A_K/K is a closed invariant subgroup of $G'_{K,k}$ and the factor group $G'_{K,k}/G_K'$ is the finite Galois group $g(K/k)$ of K/k. The Galois group G_k' of A_k/k is the factor group $G'^{a}_{K,k}$ of $G'_{K,k}$ by the closure of its commutator subgroup. The group $g(K/k)$ operates canonically on C_K, and C_k may be identified with the subgroup of C_K which consists of the $g(K/k)$-fixed elements. This identification gives rise to a canonical homomorphism of C_k' into C_K'. Furthermore, if C_K' is identified with G_K' by the canonical isomorphism, then the induced operations of $g(K/k)$ on G_K' are precisely those which are determined by the group extension $G_K' \to G'_{K,k} \to g(K/k)$.

The group-theoretical transfer homomorphism ("Verlagerung") of $G'_{K,k}$ into the subgroup G_K' induces a "reduced transfer" of G_k' $(=G'^{a}_{K,k})$ into G_K'. An analysis of a classical argument of Artin's in connection with the principal ideal theorem [Abh. Math. Sem. Univ. Hamburg **7**, 46–51 (1929)] shows the following: (A') If G_k' and G_K' are canonically identified with C_k' and C_K', the reduced transfer becomes precisely the canonical homomorphism of C_k' into C_K'. Furthermore, the following facts are evident from the definitions. (B') If k' is a field intermediate between k and K, $G'_{K,k'}$ is the inverse image in $G'_{K,k}$ of $g(K/k')$ by the natural homomorphism of $G'_{K,k}$ onto $g(K/k)$. (C') Let K'/k be a finite normal algebraic extension such that K' contains K. Then $G'_{K,k}$ can be identified with the factor group of $G'_{K',k}$

by the closure of the commutator subgroup of $G'_{K',K}$, and G_K' is thereby identified with $G^a_{K',K}$.

In the case where k is a field of algebraic functions over a finite field of constants, one has an analogous situation, except that the infinite Galois groups must be replaced by the subgroups of those automorphisms whose restrictions to the algebraic closure of the constant field are of the form $x\rightarrow x^{q^d}$, where q is the number of elements of the constant field and d is an arbitrary integer. Furthermore, one has, in this case, $C_k=C_k'$, and the above homomorphism $C_k'\rightarrow C_K'$ is now the canonical isomorphism of C_k into C_K. The above facts then contain all the fundamental theorems of the class field theory. In particular, (A′) and (B′) together yield the "translation theorem" which says that if k'/k is any finite, separable, algebraic extension the automorphism of $A_{k'}/k'$ corresponding to an element $x\epsilon C_{k'}$ induces on A_k the automorphism corresponding to the norm $N_{k'/k}(x)$. For algebraic number fields, however, (A′) and (B′) do not imply this result, because the homomorphism $C_k'\rightarrow C_K'$ is generally not one-to-one.

This comparison of the class field theories for function fields and number fields suggests the conjecture that there may be an interpretation of the idèle class group C_k for a number field from which one could derive groups $G_{K,k}$ in a natural fashion which would bear the same relationship to C_k and C_K as the Galois groups $G'_{K,k}$ do in the case of function fields, i.e., which are such that properties (A), (B), (C), analogous to (A′), (B′), (C′) above, hold for the groups $G_{K,k}$, C_k, etc., and yield the properties (A′), (B′), (C′) on passing to the factor groups $G'_{K,k}=G_{K,k}/D_K$, $C_k'=C_k/D_k$, etc.

Although such an interpretation of C_k is left open as an unsolved problem (of fundamental importance for number theory, in the opinion of the author), the group $G_{K,k}$ is defined in a canonical fashion as an extension of C_K by $g(K/k)$, and a canonical homomorphism $G_{K,k}\rightarrow G'_{K,k}$, with kernel D_K, is given, in such a way that all the above requirements are met. More precisely, the following is shown. In order that the properties of the $G_{K,k}$ contain the translation theorem, a further requirement is added to (A), (B), (C): Let x be an element of $G_{K,k}$, x' its canonical image in $G'_{K,k}$, and x'' the canonical image in G_k' of $t(x)$, where t is the transfer homomorphism of $G_{K,k}$ into C_K (it is easily seen that $t(x)$ necessarily belongs to the image of C_k in C_K). The additional requirement is then: (D) The automorphism of A_k which is induced by x' coincides with x''.

The main result is that, for each K/k, there is one and only one extension $G_{K,k}$ of C_K by $g(K/k)$, and one and only one homomorphism $G_{K,k}\rightarrow G'_{K,k}$ such that (B) and (D) are satisfied, and that then (A) and (C) are also satisfied. The proof, especially for the existence of the canonical homomorphism $G_{K,k}\rightarrow G'_{K,k}$, is long and difficult. It depends on an analysis of the group D_K (for which theorem 1, in the paper by Chevalley reviewed below, is needed) leading to a complete description of the cohomology groups for $g(K/k)$ in D_K, and thus providing the means for selecting a class of factor sets for $g(K/k)$ in C_K which defines the group extension $C_K\rightarrow G_{K,k}\rightarrow g(K/k)$ (or rather an appropriately narrow equivalence class of such group extensions) and the homomorphism $G_{K,k}\rightarrow G'_{K,k}$. The construction is accomplished by using (B) to reduce the condition (D) in a number of stages to a rather simple requirement for quadratic extensions.

These results are then applied to obtain a generalization of Artin's theory of non-abelian L-series [Abh. Hamb. 3, 89–109 (1923); 8, 292–306 (1930)], with the same bearing on the L-functions of Hecke (defined by means of characters of C_K) as Artin's theory has for the ordinary L-functions (defined by means of characters of C_K'). For this purpose, the notions of decomposition group and inertial group relative to a non-Archimedian valuation in a normal extension are amplified to yield closed subgroups H_s and H_t of $G_{K,k}$ as well as a "Frobenius class" F which is a certain coset of H_t in H_s and generates H_s. The subgroup H_t is a compact invariant subgroup of H_s, and H_s/H_t is an infinite cyclic group. H_s, H_t, and F are determined up to an inner automorphism of $G_{K,k}$ by a prime $\mathfrak{p}$ of k. Let $F_\mathfrak{p}$ and $H_\mathfrak{p}$ denote particular choices of F and H_t, respectively, belonging to the prime $\mathfrak{p}$. There is a unique invariant Haar measure of total value 1 on $H_\mathfrak{p}$, and by group translation in $G_{K,k}$ one obtains such a measure on each coset $F_\mathfrak{p}^n$, for all integers n. If φ is a continuous complex valued function on $G_{K,k}$ which is invariant under the inner automorphisms, the integral $M(\varphi, \mathfrak{p}^n)$ of φ over $F_\mathfrak{p}^n$ with respect to this measure does not depend on the particular choice of $F_\mathfrak{p}$ and $H_\mathfrak{p}$. A function $L(s, \varphi, K/k)$ of the complex variable s is now defined by the series $\log L(s, \varphi, K/k)=\sum_{\mathfrak{p},n} M(\varphi, \mathfrak{p}^n)/nN(\mathfrak{p})^{ns}$, where $N(\mathfrak{p})$ stands for the absolute norm of $\mathfrak{p}$ and the summation is extended over all positive integers n and all primes $\mathfrak{p}$ of k.

If $K=k$ and χ is a character of C_k, $L(s, \chi, k/k)$ is a Hecke L-function. On the other hand, if χ is a character of a representation of $G_{K,k}$ whose kernel contains D_K, then it may be regarded as a character of $G'_{K,k}$ and $L(s, \chi, K/k)$ is a non-abelian L-function in the sense of Artin.

Generally, these new functions are shown to have formal properties (with respect to fields intermediate between k and K and normal overextensions of K/k) exactly analogous to those of Artin's L-functions, and Artin's theory can be extended to them completely. The fact that they are meromorphic functions is proved by means of R. Brauer's theorem on induced characters [Ann. of Math. 48, 502–514 (1947); these Rev. 8, 503] by virtue of which $L(s, \chi, K/k)$ can be expressed as a product of Hecke functions.

G. Hochschild (New Haven, Conn.).

Chevalley, Claude. Deux théorèmes d'arithmétique. J. Math. Soc. Japan 3, 36–44 (1951).

Two theorems are proved here which have both been used by A. Weil in the paper reviewed above. Theorem 1: Let K be an algebraic number field of finite degree, and let E be a finitely generated subgroup of the multiplicative group of K. Let m be a positive rational integer, b an arbitrary rational integer ($\neq 0$). Then there exists a rational integer a which is prime to b and such that every element x of E which is congruent to 1 mod a (in the sense that $x-1$ is of the form ay/z, where y and z are integers of K and z is prime to a) is the mth power of an element of E.

By elementary considerations, it is shown that it suffices to show the existence of an integer a such that every element of E which is congruent to 1 mod a is the mth power of an element of K, and that this problem can be reduced to the case in which m is a power of a prime, p, and K contains a primitive mth root of unity. In this case, the author considers the Kummer extension L/K which is obtained by adjoining to K the mth roots of the elements of E. Let L_i/K, $i=1, 2, \cdots, s$, be the subextensions of L/K which are of degree p. By the "inequalities" of the class field theory, infinitely many primes of K remain prime in a given L_i so that, for each i, one can find a prime $\mathfrak{q}_i$ in K which remains prime in L_i and does not divide mb. If q_i is the rational prime in $\mathfrak{q}_i$, the least common multiple of $q_1, \cdots, q_s$ is then shown to satisfy the above requirements for a.

The second theorem expresses a group theoretical feature which is common to all class field theories, local and global. Let k be a field, Z a finite, algebraic, separable, normal extension field of k, K a field intermediate between k and Z, A/K the maximal abelian subextension of Z/K, B/k the maximal abelian subextension of Z/k, G the Galois group of Z/k, Γ the Galois group of Z/K. Assume that k is in the category of fields to which class field theory (either local or global, for algebraic numbers or algebraic functions) applies. Let R_k stand for the multiplicative group of k, in the local case, and for the idèle class group of k, in the global case. One has the reciprocity homomorphisms, or the norm residue symbols, $\psi_{A/K}$ and $\psi_{B/k}$ of R_K and R_k onto the Galois groups Γ/Γ' and G/G' of A/K and B/k, respectively. The kernels of $\psi_{A/K}$ and $\psi_{B/k}$ are the norm groups $N_{A/K}(R_A)$ and $N_{B/k}(R_B)$, respectively. Furthermore, even if Z/k is not normal, $N_{B/k}(R_B)=N_{Z/k}(R_Z)$. In addition, these homomorphisms ψ satisfy three fundamental compatibility conditions with respect to isomorphic extensions, subextensions, and translated extensions. Using only these fundamental facts, the author proves the following result. Theorem 2: Let t denote the "reduced" transfer homomorphism (Verlagerung) of G/G' into Γ/Γ'. Then, for every $x\epsilon R_k$ ($\subseteq R_K$),

$$\psi_{A/K}(x)=t(\psi_{B/k}(x)).$$

For the case of algebraic number fields, this is a classical result of Artin's [Abh. Math. Sem. Univ. Hamburg 7, 46–51 (1929)]. *G. Hochschild* (New Haven, Conn.).

Moriya, Mikao. Zur Theorie der Klassenkörper im Kleinen. J. Math. Soc. Japan 3, 195–203 (1951).

A. Examples are given of fields $\mathfrak{k}$ which do not satisfy the axiom (1) $\mathfrak{k}$ has no inseparable extensions, but which do satisfy the axiom (2) $\mathfrak{k}$ has for every positive integer n exactly one extension of degree n. B. Examples are given of fields k which are complete with respect to a discrete non-archimedean valuation and whose residue class fields $\mathfrak{k}$, though not Galois fields, satisfy both axioms (1) and (2). These fields B are valuation-completions of infinite algebraic extensions of the rational field. They have only denumerably many finite algebraic extensions, but (as is shown) their multiplicative groups contain nondenumerably many subgroups of finite index which are congruence groups. So these fields provide the example promised by Nakayama and Moriya in their papers on generalized local class field theory [see Proc. Imp. Acad. Tokyo 19, 132–137 (1943); these Rev. 7, 363, especially last sentence of the review]. (Addendum to review just cited: for references to the earlier work, mostly by M. Moriya and O. F. G. Schilling, which led to this generalized local class field theory, see Schilling's book [The theory of valuations. Math. Surveys, no. 4, Amer. Math. Soc., New York, 1950; these Rev. 13, 315].)

The existence of these fields B is implicit in earlier work of Moriya [J. Fac. Sci. Hokkaido Univ. Ser. I. 5, 9–66 (1936)]. O. F. G. Schilling [Ann. of Math. (2) 38, 551–576 (1937)] gave examples of formal power series fields which are also of the above type B. *G. Whaples.*

Kawada, Yukiyosi. On the class field theory on algebraic number fields with infinite degree. J. Math. Soc. Japan 3, 104–115 (1951).

Let k be an algebraic number field and let N^* be the infinite part of the Steinitz G-number representing its degree over the rationals. By topological methods, Kawada defines an idèle-group J_k for k and norm groups $N(K/k)$ for the finite abelian extensions K/k such that the correspondence of fields K to their norm groups gives a lattice isomorphism of the set of all abelian K/k whose degrees are prime to N^* and a certain class of subgroups of J_k which can be described entirely in terms of the structure of J_k. These results are exact analogues of those obtained by M. Moriya [J. Fac. Sci. Hokkaido Univ. Ser. I. 6, 63–101 (1937)] with ideals instead of idèles. In both cases the index of the norm group equals the degree when this degree is prime to N^*, but equals 1 when the degree divides N^*. *G. Whaples.*

✓ ***Krasner, Marc. Quelques méthodes nouvelles dans la théorie des corps valués complets.** Algèbre et Théorie des Nombres. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 24, pp. 29–39. Centre National de la Recherche Scientifique, Paris, 1950.

In this paper the author reviews his non-abelian class field theory, which was developed between 1937 and 1947. It is shown that a certain portion of the local class field theory, topological in nature, and tied in with the Hilbert ramification theory, can be generalized to arbitrary finite separable extensions of complete valued fields; there is a law of limitation for Galois extensions. In the course of this program, the author emphasizes certain fundamental principles which come to light, and which "semblent être l'expression véritable de la liaison entre l'algèbre et la métrique dans les corps valués complets". Some applications of the theory are given, including the approximation of fields of characteristic $p\neq 0$ by sequences of fields of characteristic 0. A discussion of the author's theory of analytic functions in complete valued fields is also given. The material presented, which includes a number of proofs, is contained mainly in the following papers: Mathematica, Cluj 13, 72–191 (1937); Duke Math. J. 6, 120–140; 7, 121–135 (1940); C. R. Acad. Sci. Paris 205, 1026–1028 (1937); 219, 345–347, 433–435, 473–476, 539–541 (1944); 220, 28–30, 761–763 (1945); 221, 737–739 (1945); 222, 37–40, 165–167, 363–365, 581–583, 626–628, 984–986, 1370–1372 (1946); 224, 173–175, 434–436 (1947); these Rev. 1, 260; 2, 123; 7, 363, 364, 429, 510; 8, 62, 366, 708. *B. N. Moyls* (Vancouver, B. C.).

✓ ***Krasner, Marc. Généralisations nonabéliennes de la théorie locale des corps des classes.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 71–76. Amer. Math. Soc., Providence, R. I., 1952.

Let k be a complete valuated field, and $\mathfrak{K}$ be its algebraic closure. For a rather large class of fields k, methods are outlined for determining the number of completely ramified extensions $K\subset\mathfrak{K}$ of k of fixed degree n and different $\mathfrak{d}$. The methods are based on the following fundamental principle. Let α be separable over k and let C_α be the largest circle not containing any conjugate $\alpha'\neq\alpha$ of α with respect to k. Then $\beta\epsilon C_\alpha$ implies that $k(\beta)\supseteq k(\alpha)$. The fields under discussion are those for which C_α is always the largest such circle with center α. This is always true for locally compact fields and hence for all $\mathfrak{p}$-adic fields. For $\mathfrak{p}$-adic fields the number of completely ramified extensions of degree n can be calculated explicitly. *W. H. Mills* (New Haven, Conn.).

Iyanaga, S., et Tamagawa, T. Sur la théorie du corps de classes sur le corps des nombres rationnels. J. Math. Soc. Japan 3, 220–227 (1951).

General results of the class field theory are made explicit for the case where the base field is the field R of the rational numbers.

In the first part, an explicit determination of the reciprocity homomorphism of the idèle group J of R onto the Galois group of the maximal abelian extension A/R is given. Here $J=P\times U_0\times U_\infty$, where P is the group of principal idèles, U_0 is the direct product of the local unit groups, U_p, for the finite primes p of R, and U_∞ is the multiplicative group of the positive real numbers. By Kronecker's classical result, A is obtained by adjoining all roots of unity to R. An element $u\varepsilon U_0$ defines an automorphism of A which is determined by the condition that it map every p^ν-th root ξ of unity into $\xi^{u_{p,\nu}}$, where $u_{p,\nu}\equiv u_p(\text{mod } p^\nu)$, u_p (εU_p) being the p-component of u. There results an isomorphism of U_0 onto the Galois group of A/R whose natural extension to J, for every finite abelian extension K/R, induces on each local component of J the Hasse norm residue symbol, and which, globally, yields the reciprocity homomorphism of J onto the Galois group of K/R.

In the second part, a generalization of the theory of genera (for quadratic fields) to arbitrary cyclic extensions Z/R is given. The principal genus of Z/R is defined as the subgroup H_Z of the idèle group J_Z which is constituted by the idèles $\mathfrak{a}$ for which $N(\mathfrak{a})\varepsilon PN(U_Z)$, where N denotes the norm map for Z/R, and U_Z is the group of the idèles with unit components only. By Hilbert's theorem 90 and Hasse's norm theorem, these are precisely the idèles of the form $b^{1-\sigma}uz$, where $b\varepsilon J_Z$, $u\varepsilon U_Z$, $z\varepsilon P_Z$, and σ is a generator of the Galois group of K/R. The elements of the factor group J_Z/H_Z are called the genera of Z. Their number $[J_Z:H_Z]$ is shown to be equal to the product of all the ramification indices e_p (for Z/R) of the finite primes p of R, divided by the degree of Z/R. If χ is a Chevalley differential of R which determines Z (i.e., a character of J whose kernel is $PN(J_Z)$), and $\mathfrak{a}\varepsilon J_Z$, then the local component χ_p of χ maps $N(\mathfrak{a})$ into an e_p-th root of unity. It is shown that, in this fashion, χ induces an isomorphism of the group of genera onto the group of all systems $(\xi_1, \cdots, \xi_h)$, where ξ_i is an e_{p_i}-th root of unity, $p_1, \cdots, p_h$ are the primes of R which ramify in Z, and $\xi_1\cdots\xi_h=1$. *G. Hochschild* (New Haven, Conn.).

Kuroda, Sigekatu. Über die Zerlegung rationaler Primzahlen in gewissen nicht-abelschen galoisschen Körpern. J. Math. Soc. Japan 3, 148–156 (1951).

Let R be the rational number field, and let

$$K=R(i, \sqrt{\mu}, \sqrt{\bar{\mu}}),$$

where $i=\sqrt{-1}$, is an integer of $k=R(i)$ without square divisors, neither real nor purely imaginary, and $\bar{\mu}$ is the complex conjugate of μ. Let R^* be the compositum of all the quadratic extensions $R(\sqrt{l})$, where l ranges over all the prime divisors of the discriminant of $R(\sqrt{\mu\bar{\mu}})/R$. The decomposition of rational primes p in $K^*=R^*K$ is studied.

Except when p is completely decomposed in $R^*(i, \sqrt{\mu\bar{\mu}})/R$ (which happens if and only if p belongs to the Takagi group of this abelian extension), the structure of the Galois group of K/R shows that the residual degree f_p of prime factors of p in K^*/R (though not the Artin symbol of p) is determined in an abelian way, i.e., by belonging to certain congruence classes modulo some convenient integer. If p belongs to the mentioned Takagi group (and, in this case, it splits in $R(i)$ in two prime factors π, $\bar{\pi}$), f_p is 1 or 2 according to the value of the quadratic residue symbol $[\frac{\mu}{\pi}]_2$ in $R(i)$. If c is the greatest rational integer factor of μ, and if $\mu=c\alpha$, we have $[\frac{\mu}{\pi}]_2=[\frac{c}{\pi}]_2[\frac{\alpha}{\pi}]_2$.

The author defines, for rational integers m, n, satisfying certain conditions, a biquadratic residue symbol $(\frac{m}{n})_4$. Such integers, in particular, both split into products of imaginary conjugate integers in $R(i)$: $m=\mu\bar{\mu}$ and $n=\nu\bar{\nu}$, and the author proves the reciprocity law:

$$\left(\frac{m}{n}\right)_4\left(\frac{n}{m}\right)_4=\left[\frac{\mu}{\nu}\right]_2=\left[\frac{\nu}{\mu}\right]_2.$$

The author shows that, for a prime p of the kind considered, the symbols $(\frac{\alpha\bar{\alpha}}{p})_4$ and $(\frac{p}{\alpha\bar{\alpha}})_4$ are defined (the value of $(\frac{m}{p})_4$ cannot be described by means of congruence classes mod m, because R does not contain primitive 4th roots of unity). On the other hand, $[\frac{c}{\pi}]_2$ has the same value as the ordinary quadratic residue symbol $(\frac{c}{p})_2$ in R. So, for such p, f_p is 1 if, and only if

$$\left(\frac{c}{p}\right)_2\left(\frac{\alpha\bar{\alpha}}{p}\right)_4\left(\frac{p}{\alpha\bar{\alpha}}\right)_4=+1,$$

and is 2 in the other case. *M. Krasner.*

Brauer, Richard. On the algebraic structure of group rings. J. Math. Soc. Japan 3, 237–251 (1951).

In this paper, the problem of the determination of the Schur index m over a field K (of characteristic 0) of an irreducible character χ of an arbitrary group G of finite order is reduced to the determination of Schur indices μ of suitable characters ξ of certain nilpotent subgroups of G. Moreover, these characters can be chosen, for a given χ, independently from the field K, and a method of their construction is given, which is effective in any particular case.

The Schur index m of G divides its degree, and, a fortiori, the order g of G. For determining this index, it is sufficient to determine, for each prime divisor p of g, the contribution m_p of p in m. By application of the Frobenius theorem on induced characters, it is easily proved that, if ξ is an irreducible character of some subgroup H of G such that ξ appears with a coefficient non-divisible by p in the restriction of χ to H, then $m_p=\mu_p$ (where μ is the Schur index of ξ). In particular, if the degree of ξ is a power of p, we have $m_p=\mu$.

For a given p, the author considers the groups H, which are products of some cyclic subgroup $\mathfrak{A}$ of G of order prime to p with a p-Sylow group of its normalizer $N(\mathfrak{A})$. Such groups are nilpotent, and, as is easily proved, the degrees of all their representations are powers of p. The author proves that, for a given p, a suitable group H of this kind and a suitable character ξ of H exist, which satisfy the previously indicated conditions.

The proof, which is based on the ideas of earlier papers of the author [Ann. of Math. (2) **42**, 53–61 (1941); **48**, 502–514 (1947); these Rev. **2**, 215; **8**, 503] and is the essential part of the present paper, is too technical to be outlined here. It is based, essentially, on the study of "sections" of G. A "section" is a set of classes of conjugate elements of G such that the classes (A) and (B) of 2 elements A, $B\varepsilon G$ belong to a same section if, and only if there exists a power q of p such, that A^q and B^q are conjugate. It is shown that irreducible characters of G can also be distributed, in a certain manner, in "sections" and that a one-to-one correspondence can be established between sections of classes and sections of characters such that: 1) corresponding sections have the same number of elements; 2) $\{\mathfrak{K}_1, \mathfrak{K}_2, \cdots, \mathfrak{K}_l\}$ and $\{\chi_1, \chi_2, \cdots, \chi_l\}$ being these corresponding sections, and ν being the product of the orders of centralizers of all $\mathfrak{K}_i$ $(i=1, 2, \cdots, l)$ in G, the p-adic value of the determinant $\det(\chi_i(\mathfrak{K}_j))$ is the same, as that of $\sqrt{\nu}$. For each character χ_i of this section, H is precisely the group generated by all the

classes $\mathfrak{R}_j$ of the corresponding class section. And the proof of existence of a character ξ of H with the required properties is based on the indicated property of $\det(\chi_i(\mathfrak{R}_j))$.

M. Krasner (Notre Dame, Ind.).

√★**Eichler, Martin. Arithmetics of orthogonal groups.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 65–70. Amer. Math. Soc., Providence, R. I., 1952.

Let k be an algebraic number field, R be a vector space of finite rank n over k, where a norm $|\alpha|$ of $\alpha\epsilon R$ is defined by means of a quadratic form, and S_R be a group of similarities of R, i.e. of linear transformations ξ of R, which multiply the square $|\alpha|^2$ of the norm $|\alpha|$ of each $\alpha\epsilon R$ by a non-zero constant $n(\xi)$, depending only on ξ and called norm of ξ.

Let $\mathfrak{o}$ be the ring of all integers of k. An $\mathfrak{o}$-modulus $\mathfrak{J}$ of vectors $\alpha\epsilon R$ is called (by the author) a lattice if it contains n linearly independent vectors and is generated by a finite number of vectors. Norms $n(\mathfrak{J})$ and discriminants $\mathfrak{o}(\mathfrak{J})$ of lattices $\mathfrak{J}$ are defined, in a convenient way, as certain ideals of k. If $\mathfrak{J}$ and $\mathfrak{R}$ are two lattices, the author defines the ideal $\mathfrak{J}/\mathfrak{R}$ as the set of all the $\xi\epsilon S_R$ such that $\xi\mathfrak{R}\subseteq\mathfrak{J}$. The multiplication of these ideals being defined as the multiplication of subsets of S_R, their divisibility theory appears to be quite analogous to that of ideals in a simple algebra. The part of right and left orders of an ideal of a simple algebra is played here by classes of equivalent lattices: two lattices $\mathfrak{J}$ and $\mathfrak{R}$ are called equivalent when there exists a $\xi\epsilon S_R$ such that $\mathfrak{R}=\xi\mathfrak{J}$.

In the remaining part of his paper, the author makes the assumption that k is the rational number field and that the quadratic form defining the norm $|\alpha|$ in R is definite positive. Under this assumption, there exist only a finite number of units in S_R and only a finite number of classes of equivalent lattices of R. Certain matrices and representations are defined, which describe more precisely the divisibility properties of the set of ideals of S_R. If a lattice $\mathfrak{J}$ is chosen, a vector $\beta\epsilon\mathfrak{J}$ is said to be divisible by an operator $\xi\epsilon S_R$ if there exists a $\alpha\epsilon\mathfrak{J}$ such that $\beta=\xi\alpha$. A function $\zeta(\mathfrak{J};s)$ of s, depending on lattice $\mathfrak{J}$ as parameter, can be defined as $\sum_{\alpha\epsilon\mathfrak{J}}([2n(\mathfrak{J})]^{-1}|\alpha|^2)^{-s}=\sum_{m=1}^{\infty}\delta(\mathfrak{J};m)m^{-s}$, where $\delta(\mathfrak{J};m)$ is the number of vectors $\alpha\epsilon\mathfrak{J}$ such that $|\alpha|^2=2n(\mathfrak{J})m$. In fact, this function depends only on the equivalence class j of $\mathfrak{J}$, and can be written $\zeta(j;s)$ (and $\delta(\mathfrak{J};m)$ can be written $\delta(j;m)$). The divisibility properties of vectors by operators permit one to define, starting from the previously mentioned representations, certain matricial functions $Z^{(r)}(m)$ of degree r, having the multiplicative property $Z^{(r)}(a)Z^{(r)}(b)=Z^{(r)}(ab)$ for a, b prime one to another, and these functions are linear functions of $\delta(j;m)$, where j ranges over all the equivalence classes of lattices: $Z^{(r)}(m)=\sum_j Z_j^{(r)}\delta(j;m)$. Each such function permits one to define a matricial ζ-function $\zeta^{(r)}(s)=\sum_{m=1}^{\infty}Z^{(r)}(m)m^{-s}$, and $\zeta^{(r)}(s)$ can be represented as an Euler product $\prod_p\zeta_p^{(r)}(s)$, where $\zeta_p^{(r)}(s)=\sum_{i=0}^{\infty}Z^{(r)}(p^i)p^{-is}$ is, for the prime p, the corresponding local ζ-function, obtained by means of $Z^{(r)}(m)$ if k is replaced in other definitions by its local field k_p. In certain cases these $\zeta^{(r)}(s)$ and their local factors are the same as in Hecke's theory, but, generally, they are only related in a more indirect manner to the objects of this last theory.

This paper being an address delivered at an international congress of mathematicians, no demonstrations are given. But a reader familiar enough with the topics considered is able to form a general idea about the author's methods.

M. Krasner (Notre Dame, Ind.).

Kanold, Hans-Joachim. Sätze über Kreisteilungspolynome und ihre Anwendungen auf einige zahlentheoretische Probleme. II. J. Reine Angew. Math. **188**, 129–146 (1950).

The first part of this paper was published in J. Reine Angew. Math. **187**, 169–182 (1950); these Rev. **12**, 592. The author obtains an upper bound for the νth prime q_ν of the arithmetic progression $mx+1$. In particular, he proves that $q_1<5^{\frac{1}{2}\varphi(m)}$ for $m>6$. Moreover, the following theorem is proved. Let p and q be primes and a and b arbitrary integers. Then the equation $1+q^a+q^{2a}+\cdots+q^{(p-1)a}=b^p$ is satisfied only in the cases $1+2^3=3^2$ and $1+3=2^2$. In a former paper [ibid. **184**, 116–123 (1942); these Rev. **5**, 33], the author proved that no odd perfect number N of form $p^4q_1^4q_2^2\cdots q_r^2$ exists. This was also proved independently by the reviewer [Bull. Amer. Math. Soc. **49**, 712–718 (1943); these Rev. **5**, 90]. The latter proof is much simpler in the case that $N\not\equiv 0 \pmod 3$; it uses two theorems of Nagel [Norsk Mat. Forenings Skr. Ser. I, no. **2** (1921); pp. 12–14]. For the case $N\equiv 0 \pmod 3$, the reviewer later [same Bull. **49**, 937 (1943); these Rev. **5**, 90] simplified his proof somewhat using Kanold's proof. Now, in the paper under review, the author obtains the same proof as the reviewer for the case $N\not\equiv 0 \pmod p$. But it was found independently (actually, Kanold's paper was written in 1944). Finally, the author proves that no odd perfect number of the forms $p^4q_1^6q_2^2q_3^2\cdots q_r^2$ and $p^4q_1^8q_2^2q_3^2\cdots q_r^2$ exists.

A. Brauer.

Coxeter, H. S. M. Extreme forms. Canadian J. Math. **3**, 391–441 (1951).

Let $\sum a_{ij}x^ix^j$ be a positive definite quadratic form in n variables, with determinant Δ, and let M be the minimum of the form for integer values of the variables (not all zero). The form is said to be extreme if M^n/Δ is a local maximum for variation of the coefficients a_{ij}. Suppose that the linear transformation $\xi^i=\sum c^i_jx^j$ reduces the form to the unit form, and consider the lattice in Euclidean n-space generated by the vectors $\mathbf{t}_j=\sum c^i_j\mathbf{p}_i$, where the $\mathbf{p}_i$ are the unit vectors along the axes. This lattice determines the form, and a different set of generating vectors generates an equivalent form. A form is said to be connected if it cannot be expressed as the sum of two forms involving different sets of variables, and is said to be reflexible if the corresponding point-lattice is invariant under the reflections that reverse its n basic vectors in turn. The author proceeds to enumerate the classes of connected reflexible forms, to determine their minimal vectors of the associated point-lattices, and to identify the subgroups of the lattices which keep the origin fixed with well-known finite orthogonal groups (namely those with simplicial fundamental regions). The paper concludes with a table of extreme forms related to these groups. The many interesting details of this work cannot be summarized in a brief review.

J. A. Todd (London).

Macbeath, A. M. A new sequence of minima in the geometry of numbers. Proc. Cambridge Philos. Soc. **47**, 266–273 (1951).

Consider only polynomials $f(x, y)$ of the form

$$k+px+qy-(rx+sy)^2,$$

where $|ps-qr|=1$. For every $\epsilon>0$ there is a finite set of polynomials $f_1, \cdots, f_N$ such that if f is not equivalent to a polynomial of the form f_i+c $(i=1, \cdots, N)$ then integers x, y exist such that $0\leqq f(x, y)\leqq\epsilon$. In particular, if $\epsilon=(13/8)^{1/3}$, then $N=3$ and f_1, f_2, f_3 with their corresponding "minima"

are given. This problem is related to that of finding admissible non-homogeneous lattices for $a \leqq y - x^2 \leqq b$.

L. Tornheim (Ann Arbor, Mich.).

Macbeath, A. M. The finite-volume theorem for non-homogeneous lattices. Proc. Cambridge Philos. Soc. **47**, 627–628 (1951).

If the volume of a point set K in Euclidean n-space is finite then there exists a non-homogeneous lattice, of arbitrarily small determinant, having no point in common with K. More generally, if the volume of that part of K not lying between some pair of hyperplanes is finite, the same result holds. *L. Tornheim* (Ann Arbor, Mich.).

Ollerenshaw, Kathleen. Addendum: On the critical lattices of a sphere and four-dimensional hypersphere. J. London Math. Soc. **26**, 316–318 (1951).

In this note it is proved that if a lattice is admissible for a sphere and contains three linearly independent lattice points on the boundary of the sphere then these lattice points generate the lattice. This fact is used to meet an objection to the author's determination of the critical lattices of a 4-dimensional hypersphere [same J. **24**, 190–200 (1949); these Rev. **11**, 160]. The analogous result for 4-space is shown not to be true but only when the lattice involved is critical in which case it can be generated by four suitably chosen lattice points on the surface of the hypersphere.

D. Derry (Vancouver, B. C.).

Drach, Jules. Sur la transcendance du nombre π. Bull. Sci. Math. (2) **75**, 135–145 (1951).

This is the text of a lecture given by Drach in 1892 for his degree of agrégé. The proof of trancendency and of the general theorem of Lindemann is essentially that of Weierstrass. *K. Mahler* (Manchester).

Remez, E. Ya. On series with alternating sign which may be connected with two algorithms of M. V. Ostrogradskiĭ for the approximation of irrational numbers. Uspehi Matem. Nauk (N.S.) **6**, no. 5(45), 33–42 (1951). (Russian)

Every ω in $0<\omega<1$ has an expansion

$$\omega=\frac{1}{p_0}-\frac{1}{p_0p_1}+\frac{1}{p_0p_1p_2}-\frac{1}{p_0p_1p_2p_3}+\cdots, \tag{1}$$

where $p_0, p_1, \cdots$ are integers and $0<p_0<p_1<p_2<\cdots$, and also an expansion

$$\omega=\frac{1}{q_0}-\frac{1}{q_1}+\frac{1}{q_2}-\frac{1}{q_3}+\cdots, \tag{2}$$

where $q_0, q_1, \cdots$ are integers such that $q_{n+1}\geqq q_n(q_n+1)$. If ω is irrational, these expansions are unique; if ω is rational, there is an ambiguity in the final term analogous to that for continued fractions. These expansions and related investigations have been found in the posthumous papers of M. V. Ostrogradskiĭ preserved in the State Public Library at Kiev.

The expansion (1) is obtained by the algorithm

$$1=p_n\alpha_n+\alpha_{n+1}, \quad 0\leqq\alpha_{n+1}<\alpha_n,$$

where $\alpha_0=\omega$ and $\alpha_1, \alpha_2, \cdots$ are determined in order. The expansion (2) is similarly obtained from the algorithm $q_0\cdots q_{n-1}=q_n\beta_n+\beta_{n+1}$, $0\leqq\beta_{n+1}<\beta_n$ where $\beta_0=\omega$.

The errors in taking the first $n+1$ terms of (1) and (2) as approximations to ω are $(-1)^{n+1}(p_0\cdots p_n(p_n+1))^{-1}\theta$ where $0\leqq\theta<1$ and $(-1)^{n+1}(q_n(q_n+1))^{-1}\phi$ where $0\leqq\phi<1$ respectively. If ω is a root of an algebraic equation then $p_0, p_1, p_2, \cdots$ or $q_0, q_1, \cdots$ can be obtained by transforming the equation successively into one for $\alpha_1, \alpha_2, \cdots$ or $\beta_1, \beta_2, \cdots$ respectively. Thus the root of Wallis's equation

$$x^3-2x-5=0$$

is given as $2.094553-\theta\times2.1\times10^{-6}$ $(0<\theta<1)$ and as $2.09455148132-\phi\times2.6\times10^{-10}$ $(0<\phi<1)$ by the first three terms of (1) and (2) respectively. Compare Newton's method which with three steps gives x only to eight decimal places, a supplementary investigation being required to determine the error [E. T. Whittaker and G. Robinson, The calculus of observations . . . , 2d ed., Blackie, London and Glasgow, 1926, pp. 86, 95]. *J. W. S. Cassels.*

***Hinčin, A. Ya. Cepnye drobi. [Continued Fractions].** 2d ed. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1949. 116 pp.

This is the second edition (with no substantial changes) of a work which appeared in 1936, but which is little known outside Russia. The author's name will be more familiar to most readers in the form A. Khintchine. The first two chapters contain the classical theory of continued fractions whose elements (partial quotients) are positive integers. The third and last chapter, which occupies almost half the book, is of much greater mathematical interest, since it contains a connected account of the metrical theory of Diophantine approximations, a theory which is largely the creation of Russian scholars. One of the principal theorems (proved by the author [Math. Ann. **92**, 115–125 (1924)] in 1924) is that if $f(x)$ is a positive decreasing function of x, and $\int_0^\infty x^{-1}f(x)dx$ diverges, then almost all irrational numbers α admit infinitely many rational approximations p/q satisfying $|\alpha-p/q|<q^{-2}f(q)$, whereas if the integral converges, almost no irrational numbers admit such approximations. Another interesting theorem is that of Kusmin [C. R. (Doklady) Acad. Sci. URSS **1928**, 375–380], which arose out of Gauss's statement about the measure of the real numbers α for which the remainder $r_n(\alpha)$ after n steps of the continued fraction process lies in a given interval. A proof is also given of the author's result that

$$(a_1\cdots a_n)^{1/n}\to\prod_{r=1}^{\infty}\left\{1+\frac{1}{r(r+2)}\right\}^{\frac{\log r}{\log 2}}$$

for almost all irrational numbers α in (0, 1). His deeper result concerning the existence of $\lim_{n\to\infty}(q_n)^{1/n}$ for almost all α is referred to, but not proved. *H. Davenport* (London).

ANALYSIS

Picard, H. C. Extreme value of the mean for the pth powers of n real values x. Simon Stevin **28**, 146–150 (1951). (Dutch)

The author proves the following theorem. If the mean of N real numbers is 0 and the mean of their squares is 1, then the mean of their pth powers (for any $p>2$) attains its maximum when $N-1$ of the N numbers are $-(N-1)^{-\frac{1}{2}}$ while the remaining one is $(N-1)^{\frac{1}{2}}$. *H. S. M. Coxeter.*

Nanjundiah, T. S. Inequalities relating to arithmetic and geometric means. I, II. Half-Yearly J. Mysore Univ. Sect. B., N.S. **6**, 63–77, 107–113 (1946).

Let $\mathfrak{A}(X)$, $\mathfrak{G}(X)$, $\mathfrak{H}(X)$ be the arithmetic, geometric, and harmonic means, respectively, of the set (X) of n positive elements; then, as is well known, we have

$$\mathfrak{A}(X)\geqq\mathfrak{G}(X)\geqq\mathfrak{H}(X). \tag{1}$$

For $0<a<b$, let (A), (G), (H) be the sets of n arithmetic, geometric, and harmonic means, respectively, between a and b. In part I it is shown that
$$\mathcal{H}(A)\geqq\mathcal{A}(G),\quad \mathcal{H}(G)\geqq\mathcal{A}(H).$$
Then (1) and (2) give
$$\mathcal{A}(A)\geqq\mathcal{G}(A)\geqq\mathcal{H}(A)\geqq\mathcal{A}(G)\geqq\mathcal{G}(G)\geqq\mathcal{H}(G)\geqq\mathcal{A}(H)\geqq\mathcal{G}(H)\geqq\mathcal{H}(H).$$
It is shown, further, that
$$\lim_{n\to\infty}\mathcal{H}(A)=\lim_{n\to\infty}\mathcal{A}(G)=(b-a)/[\log b-\log a];$$
and integral analogues are given.

In part II it is shown that, for positive numbers $a_1, a_2, \cdots$, and for the arithmetic and geometric means A_n and G_n, respectively, of $a_1, \cdots, a_n$, the expressions $(A_n/G_n)^n$ and $n(A_n-G_n)$ are both nondecreasing functions of n for $n\geqq 1$. It is pointed out that the inequality between the arithmetic and geometric means is an immediate consequence of either of these results; and integral analogues are established.

E. F. Beckenbach (Los Angeles, Calif.).

Lakshmanamurti, M. On the upper bound of $\sum_{i=1}^n x_i^m$ subject to the conditions $\sum x_i=0$ and $\sum x_i^2=n$. Math. Student 18, 111–116 (1950).

Let $\alpha_m=n^{-1}\sum_{i=1}^n x_i^m$ with $3\leqq m=p/q$ and p, q integers with q odd. The extrema of α_m subject to $\sum x_i=0$ and $\sum x_i^2=n$ are studied. In particular, it is shown that $n\alpha_m\leqq[(n-1)^{m-1}+(-1)^m](n-1)^{1-m/2}$ is the best upper bound, generalizing results previously proved by several authors for $m=3, 4$. It is also shown that $\alpha_{2m}\geqq\alpha^2_{m+1}+\alpha_m^2$ is (under the same restrictions) in a sense a "best" inequality (previously known for $m=2$). *J. Kiefer* (Ithaca, N. Y.).

Meulenbeld, B. Note on some theorems of Erdös and Grünwald. Nederl. Akad. Wetensch. Proc. Ser. A. 54 = Indagationes Math. 13, 329–334 (1951).

In this paper the following extensions are given of theorems of Erdös and Grünwald [Bull. Amer. Math. Soc. 46, 954–958 (1940); Ann. of Math. (2) 40, 537–548 (1939), pp. 537–540; these Rev. 2, 242; 1, 1]. I. Let $f(x)$ be a real continuous function in the interval $-a\leqq x\leqq a$ and satisfy the conditions (i) $f(a)=f(-a)=0$; $f(x)>0$ for $-a<x<a$, $\max f(x)=M$. Let $\phi(x)=f(x)/(a^2-x^2)$ and let $\phi(x)$ satisfy the inequality (ii) $\phi(x_1)\phi(x_2)\leqq\phi^2([x_1+x_2]/2)$ for each pair (x_1, x_2) with $f(x_1)=f(x_2)$ and $-a\leqq x_1\leqq x_2\leqq a$. Then the area of the curve is less than or equal to $\frac{2}{3}$, the area of the "tangential rectangle," $\int_{-a}^{a}f(x)dx\leqq 4aM/3$. Equality holds here only if $f(x)=(a^2-x^2)(M/a^2)$. II. If $f(x)$ is a real continuous function satisfying the conditions (i), if $f(x_1)=D$, $f(x_2)=E$, $(0\leqq D\leqq M;\ 0\leqq E\leqq M,\ -a\leqq x_1\leqq x_2\leqq a)$, and if $\phi(x)$ satisfies (ii), then $x_2-x_1\leqq 2a[1-(DE)^{1/2}/M]^{1/2}$. Equality occurs here only if $x_1=-x_2$, $\phi(x_1)\phi(-x_1)=\phi^2(0)=M^2/a^4$. An extension is also given of a theorem of Szekeres [cf. L. Kuipers, Nieuw Arch. Wiskunde (2) 23, 243–246 (1951); these Rev. 13, 232]. *E. Frank* (Chicago, Ill.).

Kreĭn, M. G. The ideas of P. L. Čebyšev and A. A. Markov in the theory of limiting values of integrals and their further development. Uspehi Matem. Nauk (N.S.) 6, no. 4(44), 3–120 (1951). (Russian)

This is an expository memoir, although the subject matter is given a unified treatment with many new proofs and new generality in some of the results. The problem of the title is to estimate $\int_\xi^\eta\Omega(t)d\sigma(t)$, where $\sigma(t)$ is a nondecreasing function, $\Omega(t)$ is continuous, and the first $n+1$ (generalized) moments $\int_a^b u_k(t)d\sigma(t)$ have prescribed values c_k, $\{u_k(t)\}$ being a given sequence of functions with appropriate properties, and $a\leqq\xi<\eta\leqq b$. The original problem had $\Omega(t)=1$, $u_k(t)=t^k$, and arises naturally in the theory of probability. A related problem is the restricted moment problem for $\{u_k\}$: to find necessary and sufficient conditions for a given $\{c_k\}_0^n$ to be the first $n+1$ moments of a nondecreasing function; special cases are the restricted Hausdorff, Stieltjes and Hamburger moment problems, the trigonometric moment problem, the coefficient problem for regular functions with positive real part, and the Pick-Nevanlinna interpolation problem. The first 21 pages give a detailed historical introduction.

The first and longer chapter is devoted to the two general problems just described, and the second chapter to the same problems under the restriction $0\leqq d\sigma\leqq L dt$ with a given L, or more generally $d\phi\leqq d\sigma\leqq d\psi$ with given ϕ, ψ. The methods depending on continued fractions, used in the original form of the problems, are not available in the general case; the author devotes some attention to geometrical methods, but his chief reliance is on what he calls the method of maximal mass [see below].

We are given $n+1$ continuous functions $u_k(t)$, and we call a sequence $\{c_k\}_0^n$ positive if $P(t)=\sum_0^n a_k u_k(t)\geqq 0$ implies $\sum_0^n a_k c_k\geqq 0$. The first section deals with the fundamental (known) theorem that positive sequences coincide with the moments of nondecreasing functions $\sigma(t)$; illustrations are given in terms of the special theories mentioned above. The next section deals with maximal mass: for given $\{c_k\}$ consider the set V of all $\sigma(t)$ with the c_k as moments, put $\rho(\xi)=\inf\sum a_k c_k/P(\xi)$ (over all positive $P(t)$; then $\rho(\xi)$ is the maximum over V of $\sigma(\xi+)-\sigma(\xi-)$, the "maximal mass." Section 3 is a general discussion of Chebyshev systems of functions ($P(t)$ has at most n roots if not identically zero); henceforward $\{u_k\}$ is assumed to be such a system. Sections 4 and 5 give a detailed investigation of solutions $\sigma(t)$ of the moment problem which have only a finite number of points of increase. The index is twice the number of points of increase of $\sigma(t)$, one at an endpoint counting only half a time; a solution is called canonical if of index at most $n+2$, and principal if of index $n+1$. Theorems are given on the uniqueness of solutions with maximal mass at an assigned point, and on the existence and uniqueness of principal solutions. Section 6 deals with the variation of the points of increase of $\sigma(t)$ when the masses associated with them are varied. Section 7 deals with the formula of numerical integration associated with a canonical solution of the moment problem, and the solution of the original problem of estimating an integral when $\xi=a$, $\eta=b$. Section 8 solves the general problem of estimating $\int_a^\xi\Omega(t)d\sigma(t)$ with $n+1$ given moments, following P. G. Rehtman [Zapiski Harkov. Mat. Obščestva (4) 15, 69–80 (1938)]. Sections 9 and 10 deal with the modifications necessary respectively for periodic $u_k(t)$ and for an infinite interval (a, b).

The second part of the paper [pp. 96 ff.] deals with the original problem under the restriction $d\phi\leqq d\sigma\leqq d\psi$, and with the following extremal problem (containing many special problems which have been treated by various authors): given $\{c_k\}_0^n$ and $\lambda(t)$, $\mu(t)$, continuous, with $d\mu>0$, $|d\lambda|<d\mu$, to minimize
$$\int_a^b|P(t)|d\mu(t)+\int_a^b P(t)d\lambda(t),\quad P(t)=\sum_0^n a_k u_k(t),$$
under the condition $\sum c_k a_k=1$. *R. P. Boas, Jr.*

Korenblyum, B. I. On a problem of interpolation. Doklady Akad. Nauk SSSR (N.S.) 81, 991–994 (1951). (Russian)

The author formulates in several ways and proves the following necessary and sufficient condition for there to be a solution of bounded variation of

$$\int_0^1 x^{\lambda_n} d\sigma(x) = u_n, \quad n=1, 2, \cdots,$$

under the condition $\lambda_{n+r+1}/\lambda_n \geqq k > 1$: there exists $g(x)$ having an rth derivative of bounded variation on every finite interval, satisfying $\int_0^\infty x^r |dg^{(r)}(x)| < \infty$; $g^{(i)}(x) = o(1)$ as $x \to \infty$, $i=1, \cdots, r$; $g(\lambda_n) = u_n$. For $r=0$ and $r=1$ he expresses the conditions in terms of λ_n and u_n alone; for $r=0$ they reduce to the simple form $\sum |\Delta u_n| < \infty$.

R. P. Boas, Jr. (Evanston, Ill.).

Calculus

★**Repertorio di matematiche. A cura di Mario Villa.** Cedam, Padova, 1951. xviii+731 pp. 5000 lire.

This volume is intended for the use of teachers and prospective teachers in the secondary schools, to improve their fundamental understanding of mathematics. It is a collection of twenty-one articles by fourteen different authors. Principal attention is paid to the elementary and historical point of view, though there are some references to modern researches. There is a detailed table of contents at the beginning of the volume, and a fairly extensive index at the end. Systematic bibliographical references for each chapter would have added to the value of the volume. There are only some scattered footnote references to sources. The headings and authors of the articles are as follows: Art. I. Elements of the theory of numbers, Giovanni Ricci; Art. II. Rational, real and complex numbers, Piero Buzano; Art. III. Review of algebraic analysis, Vincenzo Amato; Art. IV. Groups, Guido Zappa; Art. V. Algebraic and transcendental numbers, Giovanni Ricci; Art. VI. Functions, limits, and infinite algorithms, Gianfranco Cimmino; Art. VII. Postulates of Euclidean and non-Euclidean geometry, Fabio Conforto; Art. VIII. Point transformations, Mario Villa; Art. IX. Synthetic methods for solving problems of plane geometry, Luigi Campedelli; Art. X. Problems solvable by ruler and compass, and classical problems, Renato Calapso; Art. XI. On the theory of magnitude and equivalence, Ugo Cassina; Art. XII. Length, area, and volume, Ugo Cassina; Art. XIII. Applications of algebra to geometry, Salvatore Cherubino; Art. XIV. Review of analytic geometry and trigonometry, Mario Villa and Amedeo Agostini; Art. XV. Review of differential and integral calculus, Cesare Rimini; Art. XVI. Curves and surfaces, Luigi Campedelli; Art. XVII. Elements of the theory of analytic functions, Gianfranco Cimmino; Art. XVIII. Vectors, Piero Buzano and Cesare Rimini; Art. XIX. Approximate calculation, Cesare Rimini; Art. XX. Probability, financial and actuarial mathematics, Filippo Sibirani; Art. XXI. Notes on the history of mathematics, Amedeo Agostini.

L. M. Graves (Chicago, Ill.).

★**Gillespie, R. P. Partial differentiation.** Oliver and Boyd, Ltd., Edinburgh, London; Interscience Publishers, Inc., New York, N. Y., 1951. viii+107 pp. $1.95.

★**Guillemin, E. A. The Mathematics of Circuit Analysis.** John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1949. xiv+590 pp. $7.50.

This unusual book for undergraduate students develops the mathematical background for various phases of electrical network theory. The main subjects are determinants, matrices, linear transformations, quadratic forms, vector analysis, complex variables, Fourier series, and Fourier integrals. Topics treated which are not available in other texts include: the effect of constraints on the latent roots of a quadratic form, Hilbert transforms, Hurwitz polynomials, positive real functions, and Sommerfeld's integral. Strangely enough there are no circuit diagrams, in fact electrical networks are not mentioned. *R. J. Duffin.*

Iyengar, K. S. K., Madhava Rao, B. S., and Nanjundiah, T. S. Some trigonometrical inequalities. Half-Yearly J. Mysore Univ. Sect. B., N.S. 6, 1–12 (1945).

The authors show that the largest λ and smallest μ for which $\cos \lambda x \geqq x^{-1} \sin x \geqq \cos \mu x$ in $(0, \pi/2)$ are

$$\lambda = (2/\pi) \cos^{-1} (2/\pi) \quad \text{and} \quad \mu = 3^{-\frac{1}{2}}.$$

They give some additional related inequalities.

R. P. Boas, Jr. (Evanston, Ill.).

Emden, Karl. Eine Lösung für $\int e^{b(x+a\cos x)}dx$. Z. Angew. Math. Physik 2, 289–292 (1951).

Rutishauser, Heinz. Bemerkungen zur Arbeit von K. Emden "Eine Lösung für $\int e^{b(x+a\cos x)}dx$." Z. Angew. Math. Physik 2, 292–293 (1951).

Under certain circumstances the charge Q on a condenser with vibrating plates satisfies

$$\dot{Q} + (RC_0)^{-1}(1-a \sin \omega t)Q + k_0 = 0.$$

The formal solution of this equation involves evaluating the integral given in the title. Emden assumes that the integral can be expressed in the form

$$b^{-1}(A_0 + \sum_{n=1}^\infty (A_n \sin nx + B_n \cos nx)) \exp (b(x+a \cos x))$$

and obtains an infinite linear system of equations for the A_0, A_n and B_n whose matrix of coefficients is somewhat similar to a Jacobi matrix. This infinite system Emden solves and obtains A_0, A_n and B_n as power series in a with coefficients rational functions of b. These power series have a finite radius of convergence for a, given b, but the convergence of the expression for the integral is not discussed. Rutishauser replaces the real Fourier series in the above by a complex Fourier series which can be solved formally by continued fractions. *F. J. Murray* (New York, N. Y.).

Gross, B. Über Funktionen von Delta-Funktionen. Z. Naturforschung 6a, 676–679 (1951).

A generalization of the representation of the Dirac delta function is presented with the aid of composite functions and an additional parameter. *R. V. Churchill.*

Albertoni, S., e Cugiani, M. Sul problema del cambiamento di variabili nella teoria delle distribuzioni. Nuovo Cimento (9) 8, 874–888 (1951).

The authors study the transformation on the space of distributions (in the sense of Schwartz) induced by a point transformation on the underlying space, which is here generally the real line. Their conclusions are applied to derive rigorously some well known formulas concerning the Dirac "function". *I. E. Segal* (Princeton, N. J.).

Iwata, Giiti. Transformation functions in the complex domain. Progress Theoret. Physics 6, 524–528 (1951).

The author replaces the integral, in the usual definition of inner product, by a contour integral so that, for example, the familiar transformation function $\delta(x-x')$ is replaced by $1/(z'-z)$. He then finds eigenvalues and eigenfunctions for the operators $z\bar{z}$, $\bar{z}$ where $\bar{z}z-\rho z\bar{z}=1$, ρ being a constant.

T. E. Hull (Vancouver, B. C.).

Taylor, A. E. L'Hospital's rule. Amer. Math. Monthly **59**, 20–24 (1952).

Knudsen, H. Lottrup. A note on a vector formula. Quart. Appl. Math. **9**, 431–435 (1952).

The following formula is established

$$\int_A d\mathbf{a}\cdot[\mathbf{r}\epsilon-\epsilon\mathbf{r}]\cdot\mathbf{B}=\int_V dv[2\mathbf{B}+\mathbf{r}\cdot\nabla\mathbf{B}-\nabla\mathbf{B}\cdot\mathbf{r}],$$

where ϵ is the unit dyad, $\mathbf{B}$ is a vector field, and the integrals are taken over a closed surface A and the included volume V. For an irrotational field the last two terms on the right cancel. The author makes two applications to fields of central force.

L. M. Milne-Thomson (Greenwich).

Theory of Sets, Theory of Functions of Real Variables

Fraïssé, Roland. Conséquence d'une hypothèse précédente, et nouvelle hypothèse permettant de nommer un bon ordre du continu. C. R. Acad. Sci. Paris **233**, 342–343 (1951).

The author derives from a hypothesis "A" of earlier notes of his [same C. R. Acad. Sci. Paris **230**, 1557–1559 (1950); **232**, 1793–1795 (1951); these Rev. **12**, 14; **13**, 99] a law which assigns to each countable ordinal w a well-ordering of the natural numbers having the order-type w. A similar construction is carried out for the case of the power of the continuum

R. Arens (Los Angeles, Calif.).

Pi Calleja, Pedro. On the nondenumerability of the continuum. Revista Unión Mat. Argentina **15**, 67–69 (1951). (Spanish)

A simple proof of the nondenumerability of the continuum utilizing a covering argument which avoids decimal representations and implicit use of the axiom of choice.

T. A. Botts (Charlottesville, Va.).

Nagumo, Mitio. Characterization of the linear continuum. J. Sci. Gakugei Fac. Tokushima Univ. **1**, 7–9 (1950). (Esperanto)

A linear continuum means a simply ordered system that is isomorphic to the system of the real numbers. In this note we wish to show that the linear continuum may be characterized as a continuous simply ordered system with a simply transitive group of automorphisms.

Extract from paper.

Kunugui, Kinjiro. Sur une généralisation de la coupure de Dedekind. J. Math. Soc. Japan **3**, 232–236 (1951).

MacNeille [Trans. Amer. Math. Soc. **42**, 416–460 (1937)] generalized the notion of Dedekind cut to partially ordered sets, so that the collection of all cuts forms a complete lattice K in which the partially ordered set R is imbedded. The author shows by an example that even if R is dense, its lattice K of MacNeille cuts need not be continuous, as the set of Dedekind cuts always is. To say that K is continuous means that for every cut A_1, A_2 in K, either A_1 has a greatest element or A_2 has a least element. The author modifies the MacNeille definition of cut so that the collection of cuts (in the new sense) of a dense ordered set is always continuous, as follows. A pair of non-empty sets A_1, A_2 of a partially ordered set R is called a cut if (1) every element of A_1 precedes every element of A_2; (2) A_1 and A_2 are directed sets; (3) A_1 and A_2 are maximal relative to (1) and (2). The question whether the collection of all cuts in this new sense is necessarily a complete lattice is not treated. Presumably it need not be.

O. Frink (State College, Pa.).

Monteiro, A. A., et Peixoto, M. M. Le nombre de Lebesgue et la continuité uniforme. Portugaliae Math. **10**, 105–113 (1951).

A positive number r is termed a Lebesgue number for an open covering $\{E_i\}_{i\epsilon I}$ of a metric space E provided for each $x\epsilon E$ there is an $i\epsilon I$ such that the open spherical neighborhood about x of radius r is contained in E_i. For a metric space E several properties, principally the following four, are shown to be equivalent: (1) every open covering of E by two sets has a Lebesgue number; (1') every open covering of E has a Lebesgue number; (2) all the bounded, continuous, real-valued functions on E are uniformly continuous on E; (2') for every metric space E', all the continuous functions on E to E' are uniformly continuous on E.

T. A. Botts.

Chernoff, Herman. An extension of a result of Liapounoff on the range of a vector measure. Proc. Amer. Math. Soc. **2**, 722–726 (1951).

If μ_{it}, $i=1,\cdots,k$; $t=1,\cdots,n$ are finite, countably additive measures over a σ-field $\mathcal{S}$ of subsets of a space X, and $\mathcal{P}$ is the class of partitions $P=(E_1,\cdots,E_k)$ of X into disjoint sets $E_i\epsilon\mathcal{S}$ with $\bigcup_1^k E_i=X$, the range of the nk-dimensional vector $v(P)=\{\mu_{it}(E_i)\}$ as P varies over $\mathcal{P}$ is closed and, if the μ_{it} are non-atomic, convex. The case $k=1$ is the Lyapunov theorem referred to in the title, and the proof makes use of this result. Related results have been obtained by Dvoretzky, Wald and Wolfowitz [Pacific J. Math. **1**, 59–74 (1951); these Rev. **13**, 331] and the reviewer [Proc. Amer. Math. Soc. **2**, 390–395 (1951); these Rev. **12**, 810].

D. Blackwell (Washington, D. C.).

Dieudonné, Jean. Sur le théorème de Lebesgue-Nikodym. IV. J. Indian Math. Soc. (N.S.) **15**, 77–86 (1951).

Let μ be a measure (positive) defined on a σ-field of subsets of E such that E is measurable and has finite measure. Let $L^1=L^1(E,\mu)$, $(L^\infty=L^\infty(E,\mu))$ be, as usual, the Banach space whose elements are classes of functions summable on E (essentially bounded on E). For every function f summable on E (essentially bounded on E) the symbol $\tilde{f}$ designates the corresponding element of L^1 (L^∞), and $N_1(f)$ $(N_\infty(f))$ its norm. Let F be a Banach space, F' its conjugate, and g a mapping from E to F'. The function g is called weakly measurable in case for every a in F the numerical function $\langle a, g(t)\rangle$ is measurable on E. The reviewer and Pettis [Trans. Amer. Math. Soc. **47**, 323–392 (1940); these Rev. **1**, 338] have shown that, if F is separable, every continuous linear map of L^1 into F' is of the form $\tilde{f}=\int f(t)g(t)d\mu$ where g on E to F' is weakly measurable and bounded in norm. The above integral is the weak integral as defined by Pettis, i.e., it is the point in F' uniquely determined by the condition that $\langle x, \int f(t)g(t)d\mu\rangle=\int f(t)\langle x, g(t)\rangle d\mu$ for all x in F. In the present paper a penetrating analysis of this theorem is made. It is shown that the theorem remains valid without the assump-

tion of separability provided the following statement is true. There is a linear map $\tilde{g} \to g^*$ of the space L^∞ into the space of functions bounded on E such that g^* is in the class g and $|g^*(t)| \leqq N_\infty(\tilde{g})$ for all t in E. This statement is valid (as has been proved by J. von Neumann) if μ is Lebesgue measure on the interval $[0, 1]$. It is also valid as is shown in the present paper in case $L^1(E, \mu)$ is separable. Thus in the Dunford-Pettis result the separability of F may be replaced by that of L^1. This restatement is, of course, a corollary of the original but the analysis of the problem as presented in the present paper emphasizes the fact that the validity of the theorem does not depend upon the space F but upon the properties of the space $L^\infty(E, \mu)$.

N. Dunford (New Haven, Conn.).

Dieudonné, Jean. Sur le théorème de Lebesgue-Nikodym. V. Canadian J. Math. 3, 129–139 (1951).

Using the notation of the preceding review, consider the case where E is a compact space and μ a Radon measure. The measure μ and a function f on E integrable with respect to μ determine a continuous linear map $g \to \int gf d\mu$ defined on the Banach space $C = C(E)$ (of continuous numerical functions defined on E) with norm $\|f\| = \sup_{x \in E} |f(x)|$. If one uses the term "measure" for a continuous linear functional on the space C the Lebesgue-Nikodym theorem characterizes those "measures" which are of the form $g \to \int gf d\mu$ by a condition of absolute continuity with respect to μ. Similarly, a "vector measure" is defined to be a continuous linear map m on C to F', the conjugate of the B-space F. The problem of the present paper is that of characterizing those vector measures m for which $m(f) = \int gf d\mu$, where g is a function on E to F' and weakly integrable with respect to μ. The result is as follows: Let F be separable. Then if m is weakly absolutely continuous with respect to μ it will be of the form $\int gf d\mu$ where g on E to F' is weakly integrable if and only if for every $\epsilon > 0$ there is a compact set $K \subset E$ and a constant a_K such that $\mu(E-K) \leqq \epsilon$ and such that for every $f \epsilon L^\infty$ which vanishes on $E-K$ we have $(m(f)) \leqq a_K N_1(f)$. The function g is uniquely determined except for a set of measure zero. Using this result in the case where L^1 and F are both separable, the Banach space conjugate to $L_F{}^p$ is determined. The symbol $L_F{}^p$ is used here for the Banach space of functions f on E to F which are strongly measurable and for which the pth power of the norm is integrable with respect to μ.

N. Dunford (New Haven, Conn.).

√***Tolstov, G. P. On the curvilinear and iterated integral.** Trudy Mat. Inst. Steklov. 35, 102 pp. (1950). (Russian)

This paper is concerned with existence and properties of iterated integrals and line integrals for functions of two variables when the fundamental one-variable integral used is that of Denjoy-Khinchin [S. Saks, Theory of the integral, Warsaw-Lvov, 1937, chapter 8]. Applications are made to mixed partial derivatives, total differentials, and to the Cauchy-Riemann equations and Morera's theorem in analytic function theory. The main problem is that of equality of iterated integrals over all sub-regions of certain types of a fixed rectangle when the hypothesis of Fubini's theorem, that $|f|$ is summable over the rectangle as a function of two variables, is not satisfied.

Chapter I outlines the necessary properties of the D-K integral, gives the notation to be used, and defines, for each set E contained in a fixed rectangle R: $a \leqq x \leqq b$, $c \leqq y \leqq d$, and for each function f defined on E, the integral $\int_E dx \int f dy$ to exist when $\varphi(x) = \int_c^d f(x, y) f_E(x, y) dy$, where f_E is the characteristic function of E, exists (in the D-K sense) and is itself integrable in the same sense. If $\int_E dx \int f dy$ and $\int_E dy \int f dx$ both exist and are equal, then f is called iterated integrable and the common value is denoted by $\int_E \int f dx dy$.

Chapter II shows that an absolutely continuous change of variable does not change the value of a definite D-K integral. This is applied in chapter III to change of variable in a line integral $\int_C P dx + Q dy$, where the line integral is defined by a D-K integral with respect to arc length s along C as parameter. Theorem 2. Suppose $F(x, y)$ continuous along the rectifiable curve C: $x = \xi(s)$, $y = \eta(s)$, $0 \leqq s \leqq L$, and that a) $P = \partial F/\partial x$ and $Q = \partial F/\partial y$ exist and are finite for almost all $s \epsilon [0, L]$, and b) the derivative numbers of F with respect to x and y are finite along C except perhaps for a countable set of points. Then

$$F(x, y) = F(x_0, y_0) + \int_\Gamma P dx + Q dy,$$

where (x_0, y_0) is a fixed point on the curve C and Γ is the part of C between (x_0, y_0) and (x, y).

The next chapter gives a counter-example to an assertion of Montel [C. R. Acad. Sci. Paris 156, 1820–1822 (1913)] that if P and Q are bounded and have continuous partial derivatives $\partial P/\partial y$ and $\partial Q/\partial x$ which are almost everywhere equal, then $P dx + Q dy$ is a total differential. In the positive direction Theorem 3 gives rather weak sufficient conditions for the existence of continuous F such that $\partial F/\partial x = P$ and $\partial F/\partial y = Q$ for all points of a simply connected region G, and applies this with Morera's theorem to prove another proposition stated by Montel: Theorem 6. Let $f(z) = u + iv$ be a function of a complex variable $z = x + iy$ bounded in G. If all four first partial derivatives of u and v exist (finite) everywhere in G and satisfy the Cauchy-Riemann equations almost everywhere in G, then f is analytic. (Note f not assumed continuous here as in Menchoff's theorem [Saks, op. cit., p. 199]. Certain possible weakenings of the hypotheses are noted in this paper.)

Chapter V considers iterated integrability over every sub-rectangle R' (with sides parallel to the coordinate axes) of the fundamental rectangle R, and the related function J defined by $J(x, y) = \int_{x_0}^x d\xi \int_{y_0}^y f d\eta = \int_{y_0}^y d\eta \int_{x_0}^x f d\xi$. An example of an f is given for which the corresponding function J is discontinuous almost everywhere in R, while the integrals are Lebesgue integrals. It is also shown that if $f = \partial^2 F/\partial x \partial y = \partial^2 F/\partial y \partial x$ for all x, y in R, that is, if f is precisely a permuting mixed second partial derivative, then f is iterated integrable with variable upper limits; that is, $\int_{R'} dx \int f dy = \int_{R'} dy \int f dx$ for each rectangle R'.

Chapter VI presents an example of a function f which is a permuting mixed second partial derivative, therefore iterated integrable over each R' in R, but not summable over any R' contained in R. This example shows that no two-dimensional process depending, as the D-K integral does in one dimension, on summability on a dense set of sub-rectangles can possibly yield the primitive F of this f.

Chapter VII considers a function f defined in R, and the two partial integrals

$$P(x, y) = \int_c^y f(x, \eta) d\eta, \quad Q(x, y) = \int_a^x f(\xi, y) d\xi.$$

Theorem 7. In order that f be iterated integrable over every sub-rectangle R' (sides parallel to axes) it is necessary and sufficient that

$$\int_C P dx + Q dy = 0$$

for every contour C bounding such an R'. Theorem 8. If P and Q are continuous in each variable and are bounded (or have summable majorants) and if everywhere, with at most countably many exceptions, they have finite first derivates with respect to x and y, then f is iterated integrable over every sub-rectangle R' contained in R. Theorem 9. If P and Q are bounded and are absolutely continuous in each variable, the same conclusion holds. Theorem 10. If u and v are bounded and linearly absolutely continuous in R and if almost everywhere $\partial u/\partial y = \partial v/\partial x$, then there exists a continuous F for which $\partial F/\partial x = u$, $\partial F/\partial y = v$. Theorem 11 (generalization of Morera's theorem). If $f(z)$ is measurable in the region G and if, for each closed and convex contour C lying in G, $\int_C f(z)dz = 0$, where the integral is taken in the sense of Lebesgue, then f is equal almost everywhere in G to an analytic function.

Chapter VIII considers iterated integrability over a larger class of sub-regions of R. A curve C is called monotone if it has a parametric representation in which both functions are monotone. A curve C is suitable (pravilnyĭ) if it is simple and can be decomposed into a finite number of monotone pieces. A region G contained in R is suitable if its boundary is a suitable curve. Theorem 12 (Green's theorem). In order that $\int_G dx \int f dy$ exist for every suitable region G in R it is necessary and sufficient that $\int_C P dx$ exist, where C is the (suitable) contour bounding G. When these exist they are related by $\int_G dx \int f dy = -\int_C P dx$. This and the corresponding result with x and y interchanged yield Theorem 14. In order that f be iterated integrable over every suitable subregion G of R it is necessary and sufficient that a) $\int_C Pdx + Qdy = 0$ for every suitable closed contour C in R, and b) $\int_\Gamma P dx$ and $\int_\Gamma Q dy$ exist (independently) for each suitable curve Γ in R. As a corollary of this theorem it follows that the function J of chapter V satisfies $J(x, y) = J(x_0, y_0) + \int_\Gamma Pdx + Qdy$, where Γ is an arbitrary suitable curve in R connecting (x_0, y_0) to (x, y); hence J is continuous in both variables; this shows that the example of chapter V is iterated integrable over every sub-rectangle R' of R but not over every suitable sub-region of R. Pursuing this train of thought farther leads to theorem 15. If f is measurable in R and if $\int_G dx \int f dy$ exists for every region G in R, then f is summable in R.

In chapter IX rectifiable contours are used. Theorem 17. If f is defined and measurable in R and if there is a constant M such that $|P(x, y)| < M$ for all y and almost all x, then for each rectifiable simple closed curve C bounding a region G

$$\int_G dx \int f dy = -\int_C P dx.$$

Theorem 19. If f is iterated integrable over every sub-rectangle of R and if P and Q are bounded (P for almost all x, Q for almost all y), then f is iterated integrable in each domain G in R which is bounded by a simple rectifiable curve, and $\int_C Pdx + Qdy = 0$. Also the corollary of theorem 14 holds with Γ rectifiable. An example shows that summability of f in R is not a sufficient condition for this conclusion.

M. M. Day (Urbana, Ill.).

Sargent, W. L. C. On the integrability of a product. II. J. London Math. Soc. 26, 278–285 (1951).

Let (C_nP) denote the space of functions integrable in the (C_nP) sense (Cesàro-Perron sense of order n) on $(0, 1)$, n a positive integer or zero. If $\theta(t)$, $a < t < b$, is an nth indefinite integral of a function of bounded variation in (a, b), $\theta(t)$ is said to be BV_n in (a, b). If $k(t)$ is equivalent to a function $\theta(t)$ which is BV_n in (a, b) then $k(t)$ is said to be EBV_n in (a, b). If $k(t)$ is EBV then the symbol for the essential variation of $k(t)$ is $\int_a^b |d_e k(t)|$, and is the lower bound of $\int_a^b |d\theta(t)|$ for all functions $\theta(t)$ which are of bounded variation in (a, b) and equivalent to $k(t)$. The following theorems are proved. Theorem 1. In order that $x(t)k(t)\epsilon(C_nP)$ whenever $x(t)\epsilon(C_nP)$, it is necessary and sufficient that $k(t)$ be equivalent to a function $\theta(t)$ such that (i) the set of points G of $[0, 1]$ throughout no neighbourhood of which $\theta(t)$ is BV_n is finite, (ii) if the interval $[c, d]$ is contained in $[0, 1]$ and contains no point of G except perhaps c, then

$$\int_{\to c}^{d} (t-c)^n |d_e\theta^{(n)}(t)| < \infty.$$

Theorem 2. In order that $x(t)k(t)\epsilon(C_nL)$ whenever $x(t)\epsilon(C_nL)$, it is necessary and sufficient that there should exist a positive number δ $(0 < \delta \leqq 1)$ and a function $\theta(t)$ equivalent to $k(t)$ and such that (i) $\theta(t)$ is measurable and bounded in $(\delta, 1)$, (ii) $\theta(t)$ is BV_n in (γ, δ) whenever $0 < \gamma < \delta$, and

$$\int_{\to 0}^{\delta} t^n |d_e\theta^{(n)}(t)| < \infty.$$

R. L. Jeffery (Kingston, Ont.).

√ ***Jeffery, R. L. Non-absolutely convergent integrals.** Proc. Second Canadian Math. Congress, Vancouver, 1949, pp. 93–145. University of Toronto Press, Toronto, 1951. $6.00.

Cet exposé à tendance didactique semble permettre à qui possède l'integrale de Lebesgue d'acquérir rapidement des idées claires et générales sur les intégrales de Denjoy et leurs généralisations, en se bornant au cas d'une seule variable réelle et de la mesure ordinaire. De nombreuses démonstrations sont exposées en entier. Voici le contenu du mémoire par paragraphes:

1. Relations entre séries et intégrales non absolument convergentes. Premier aperçu sur le procédé de Denjoy permettant de passer de $F'(n)$ à $F(x)$.

2. Opérations permettant d'obtenir $F(b) - F(a)$ quand $F'(x)$ est finie. Enoncé des trois théorèmes sur lesquels est basé le processus transfini de Denjoy.

3. Intégrale de Denjoy particulière D_* obtenue en appliquant le processus ci-dessus à une fonction $f(x)$ qui n'est pas nécessairement la dérivée finie d'une fonction continue.

4. Intégrale de Perron $P(f)$ définie au moyen des fonctions φ et ψ dont les dérivées supérieure et inférieure encadrent f. Les intégrales D_* et P sont équivalentes (Alexandroff-Looman).

5. Une extension du processus transfini de Denjoy conduit à l'intégrale de Young, et à l'intégrale générale de Denjoy (ou l'intégrale de Denjoy-Khintchine) D_g. La dérivée approximative de $D_g(f)$ est égale presque partout à f.

6. Définitions descriptives. Ici, les démonstrations, beaucoup plus courtes, sont donées tout au long. Si $F'(x) = f(x)$, F est l'intégrale de Newton de f. Si l'on a presque partout $F'(x) = f(x)$, f étant p.p. finie et mesurable, et F étant absolument continue, F est l'intégrale de Lebesgue de f. Des définitions analogues, dites descriptives, pour les intégrales de Denjoy, sont dues à Lusin et Saks. Ils introduisent les fonctions "absolument continues généralisées au sens restreint" (ACG*). Si l'on a p.p. $F'(x) = f(x)$, F étant (ACG*), alors $F = D_*(f)$. Quelques indications sont données sur la démonstration de l'équivalence des deux définitions de D_*.

La formule d'intégration par parties pour D_s est démontrée d'après Zygmund. L'intégrale de Young et l'intégrale D_g sont susceptibles de définitions analogues, la continuité (ACG*) étant remplacée par une notion voisine, la continuité (ACG). Les démonstrations sont les mêmes.

7. Ridder a donné de l'intégrale D_g une définition analogue à celle de Perron. La démonstration de l'équivalence, qui est assez longue, est exposée ici en entier.

8. La "dérivée de Cesàro" d'une fonction dont l'intégrale D_s existe se définit par un procédé rappelant celui de la sommation C_1 d'une série. L'intégrale de Cesàro-Perron C_1 a été définie par Ridder à partir de cette dérivée comme l'intégrale de Perron l'est à partir de la dérivée ordinaire. De nombreuses propriétés de l'intégrale C_1 sont démontrées, y compris une formule d'intégration par parties.

9. Différentes généralisations de l'intégrale C_1 ont été données par Burkill, Miss Sargent, Jeffery, et Ellis. La plus intéréssante parait être l'intégrale M_r de Ellis (r entier >0) qui admet une définition par induction sur r analogue à la définition descriptive de l'intégrale de Denjoy. Plusieurs propriétés de cette intégrale sont indiquées, ainsi qu'un résultat de Ellis non encore publié sur les fonctions vérifiant la propriété de Darboux.

10. Le problème du calcul des coefficients d'une série trigonométrique convergeant vers une fonction $f(x)$ à partir de f a été résolu par Denjoy. Des indications sur le procédé employé sont données. Plusieurs types d'intégrales ont été introduits en vue de résoudre directement ce problème (Verblunsky [Fund. Math. **23**, 193–236 (1934)], Marcinkiewicz et Zygmund [ibid. **26**, 1–43 (1936)], James et Gage [Trans. Roy. Soc. Canada. Sect. III. (3) **40**, 25–35 (1946); Canadian J. Math. **2**, 297–306 (1950); ces Rev. **9**, 19; **12**, 94]). Les définitions en sont indiquées. Ces types comprennent l'intégrale C_1, mais sont plus généraux, sauf peut-être le deuxième. *R. de Possel* (Alger).

Szász, Otto, and Todd, John. Convergence of Cauchy-Riemann sums to Cauchy-Riemann integrals. J. Research Nat. Bur. Standards **47**, 191–196 (1951).

The paper deals with conditions for the truth of the formula

$$(*)\qquad \lim_{h\to+0} h\sum_{\nu=1}^{\infty} f(\nu h)=\int_0^\infty f(x)dx,$$

where the integral is taken in the Cauchy-Riemann sense, i.e. as the limit of the integral over (ϵ,ω) when $\epsilon\to0$, $\omega\to\infty$. The main results are that the following sets of conditions are sufficient:

$$(1)\quad f(x)=\frac{1}{x}\sum_1^\infty b_n \sin nx,\quad \sum_1^\infty |b_n|<\infty,\quad \int_{\to0}^{\pi} f(x)dx \text{ exists;}$$

$$(2)\quad f(x)=\frac{1}{x}\int_0^\infty \eta(t)\sin xt\,dt,\quad \int_0^\infty|\eta(t)|\,dt \text{ finite.}$$

The proofs are by direct substitution of $f(x)$ into the left hand side of (*) followed by the use of special Fourier expansions and justification of the relevant limit processes. Some examples involving Bessel functions are discussed independently in the concluding sections. [Note by reviewer: The last condition in (1) seems redundant, and seems in fact to be established as a consequence of the other hypotheses by the uniform convergence of the series in (15) on p. 193.]
A. E. Ingham (Cambridge, England).

Froda, Alexandru. Introduction to the study of measurability of multiform and uniform functions of a real variable. Acad. Repub. Pop. Române. Bul. Şti. A. **1**, 197–204 (1949). (Romanian. Russian and French summaries).

Froda, Alexandru. Measurability of multiform and uniform functions of a real variable. Acad. Repub. Pop. Române. Bul. Şti. A. **1**, 835–846 (1949). (Romanian. Russian and French summaries).

Froda, Alexandru. Operations on measurable multiform and uniform functions. Acad. Repub. Pop. Române. Bul. Şti. A. **1**, 937–945 (1949). (Romanian. Russian and French summaries).

Let Δ_n be an n-dimensional interval, and f a single- or multiple-valued real function defined on a measurable set E of positive measure contained in Δ_n. Let $f(P)$ denote the set of all values assumed by f at the point $P\epsilon E$. If $f(P)$ is closed for all $P\epsilon E$, then f is said to be vertically closed. For a set Γ of real numbers, $E[f;\Gamma]$ is the set of all $P\epsilon E$ such that $f(P)\subset\Gamma$. It is shown that if $E[f;\Gamma]$ is measurable for every Γ of the form $\alpha\leqq x\leqq\beta$ $(-\infty\leqq\alpha\leqq\beta\leqq+\infty)$, then $E[f;\Gamma]$ is measurable for all half-infinite open or closed intervals. Such functions are said to be measurable in support. If f is vertically closed, then general α and β can be replaced by rational α and β in the theorem just stated. If $f_1,\cdots,f_m$ are functions of the type described, let the sum (product) of $f_1,\cdots,f_m$ be the function such that $f(P)$ is the union (intersection) of the sets $f_1(P),\cdots,f_m(P)$. The sum of a finite set of functions measurable in support is also measurable in support, but their product need not be. Limiting processes for sequences of functions measurable in support are also studied. *E. Hewitt* (Seattle, Wash.).

Yih, Chia-shun. An extension of Dehn's theorem on the approximation of a function by a power series. Math. Student **18**, 117–122 (1950).

Let $f(x,y)$ be defined in the neighborhood of (a,b) and let n be a fixed positive integer. This note deals with necessary and sufficient conditions for the existence of constants a_{ij} $(i,j=0,1,\cdots,n)$ and functions $\epsilon_{lm}(h,k)$ $(l+m=n)$ such that

$$f(a+h,b+k)=\sum_{i+j\leqq n-1} a_{ij}h^ik^j+\sum_{l+m=n}[a_{lm}+\epsilon_{lm}(h,k)]h^lk^m$$

and $\epsilon_{lm}(h,k)\to0$ as $(h,k)\to(0,0)$. The conditions are stated in terms of the existence of limits for certain difference quotients of the function f. Derivatives of order higher than the first need not exist. The corresponding results for functions of one variable are due to Max Dehn [Math. Student **15**, 79–82 (1949); these Rev. **10**, 690]. *A. E. Taylor.*

Dyson, F. J. Continuous functions defined on spheres. Ann. of Math. (2) **54**, 534–536 (1951).

The author proves that, for any real-valued continuous function $f(x)$ defined on a two-sphere S^2, there exist four points x_1, x_2, x_3, x_4 on S^2 forming the vertices of a square on a great circle of S^2 (i.e. x_1, x_2, x_3, x_4 are the end points of two mutually orthogonal diameters of S^2) such that $f(x_1)=f(x_2)=f(x_3)=f(x_4)$. The proof is ingenious and is based on the following lemma: Let L be a closed bounded set in a three space R^3 not containing the origin Z of R^3 such that every continuous curve joining Z with point at infinity intersects L. If the intersection D of L and its reflection L' with respect to Z is a continuum, then there exists two points x_1 and x_2 in D equidistant from Z and sustaining a

right angle at Z. Generalizations of this result to higher dimensional cases, in which S^2 is replaced by an n-sphere S^n ($n>2$) and four points x_1, x_2, x_3, x_4 are replaced by $2n$ points on S^n which are the end points of n mutually orthogonal diameters of S^n, are still open. *S. Kakutani.*

Landis, E. M. On functions representable as the difference of two convex functions. Doklady Akad. Nauk SSSR (N.S.) 80, 9–11 (1951). (Russian)

Let $f(x, y)$ be the difference of two (continuous) convex functions, defined on a square in the (x, y)-plane. The author's main result (Theorem 3) is that if Ω is the set of points at which the total differential of f exists and equals 0 then the values assumed by f at points in Ω form a set of zero measure. This is based on his Theorem 1, which asserts that the partial derivatives f_x' and f_y' are functions of bounded planar variation [cf. A. S. Kronrod, same Doklady 66, 797–800 (1949); these Rev. 11, 19]. Proofs are given in outline. *H. P. Mulholland* (Birmingham).

Mathis, H. F. The extension of a rectangular matrix of continuous functions. Math. Mag. 25, 3–6 (1951).

L'autore dimostra che se le funzioni

$$a_{i,j}(x, y) \quad (i=1, \cdots, p;\ j=1, \cdots, q;\ p<q)$$

sono continue in una regione R, piana, chiusa e limitata, con la frontiera costituita da un numero finito di curve semplici e chiuse, e se il rango della matrice $\|a_{i,j}(x, y)\|$ è sempre uguale a p in R, a questa matrice si possono aggiungere $q-p$ righe, i cui elementi siano polinomi, in guisa che la matrice quadrata risultante sia non singolare in ogni punto di R. Il teorema vale anche per funzioni di n variabili, date in una n-cella; la cosa è stata già dimostrata da Ważewski [Compositio Math. 2, 63–68 (1935)] e ritrovata dall'autore [Proc. Amer. Math. Soc. 1, 344–345 (1950); questi Rev. 12, 168]. *G. Scorza Dragoni* (Padova).

√*Cesari, L., and Radó, T. **Applications of area theory in analysis.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 174–179. Amer. Math. Soc., Providence, R. I., 1952.

This paper is a summary of an invited address given at the 1950 International Congress of Mathematicians. The purpose of the paper is to describe the two-dimensional concepts of bounded variation and absolute continuity devised by L. Cesari, T. Radó, and P. V. Reichelderfer and to show applications of these concepts in the transformation of double integrals, calculus of variations, and Lebesgue area theory. *R. G. Helsel* (Columbus, Ohio).

Theory of Functions of Complex Variables

√*Bieberbach, L. **Einführung in die Funktionentheorie.** 2d ed. Verlag für Wissenschaft und Fachbuch, Bielefeld, 1952. 220 pp. 12.60 DM.

This is an introductory textbook, oriented somewhat in the direction of applications, but more detailed and rigorous and covering more ground than similar books in English. Account is frequently taken of the most recent proofs of standard theorems. There is a chapter on practical conformal mapping, including a brief account of Bergman's use of orthogonal functions. *R. P. Boas, Jr.*

*Markovitch, D. **Sur le théorème de Grace.** Premier Congrès des Mathématiciens et Physiciens de la R.P.F.Y., 1949. Vol. II, Communications et Exposés Scientifiques, pp. 67–71. Naučna Knjiga, Belgrade, 1951. (Serbo-Croatian. French summary)

The author gives a new proof of the theorem of Grace [Proc. Cambridge Philos. Soc. 11, 352–357 (1902)] and uses this to prove that for each fixed ν, $\nu=1, 2, \cdots, n$, every circle containing α and

$$\beta_k=\alpha-e^{2k\pi i/\nu}\left[\frac{n!P(\alpha)}{(n-\nu)!P^{(\nu)}(\alpha)}\right]^{1/\nu}, \quad k=1, 2, \cdots, n,$$

contains at least one root of the nth degree polynomial $P(z)$. When $\nu=1$ this gives Laguerre's theorem.

A. W. Goodman (Lexington, Ky.).

Walsh, J. L. Note on the location of the critical points of a real rational function. Proc. Amer. Math. Soc. 2, 682–685 (1951).

In the present note the author derives some new results which supplement the theorems of section 5.8.3 of his book [The location of critical points . . . , Amer. Math. Soc. Colloq. Publ. vol. 34, New York, 1950; these Rev. 12, 249]. These include a theorem concerning a real rational function $p(z)$ of degree $n=-2m>0$, which has all its poles interior to the unit circle C: $|z|=1$, which has a real k-fold zero $(0<k<n)$ at x_1 interior to C and which has at infinity a zero of multiplicity $n-k$. All the nonreal finite critical points of $p(z)$ exterior to C are found to lie in the closed interior of the circle $(n-k)|z-x_1|^2=k(1-x_1^2)$. *M. Marden.*

Elliott, H. Margaret. On approximation to functions satisfying a generalized continuity condition. Trans. Amer. Math. Soc. 71, 1–23 (1951).

The author continues work which was started by Walsh and was extended by Walsh, Sewell and herself. Let C be a finite set of mutually exterior rectifiable Jordan curves in the z-plane, $\bar{C}$ the sum of their closed interiors, s the arc-length, and let $u(z)$ be harmonic in C and continuous on $\bar{C}$, $v(z)$ conjugate to $u(z)$ and $f(z)=u(z)+iv(z)$, $\pi_n(z)$ a polynomial of degree n and $p_n(z)=\Re\pi_n$. Let $w=\phi(z)$ map the exterior of C on $|w|>1$ so that $\phi(\infty)=\infty$, C_R be the image of $|w|=R$ $(R>1)$. J. L. Walsh, W. E. Sewell and the author [Trans. Amer. Math. Soc. 67, 381–420 (1949); these Rev. 11, 515; cf. Walsh and Elliott, ibid. 68, 183–203 (1950); these Rev. 11, 315] have discussed relations between continuity properties of $u(z)$ on the curves C or C_R ("problem α" or "β", respectively) and the degree of convergence of sequences $p_n(z)$ approximating to $u(z)$, $\pi_n(z)$ to $f(z)$, on $\bar{C}$. The continuity conditions were of the Lipschitz type. Instead, weaker conditions are used here: the continuity modulus $\omega(\delta)$ $(\delta>0)$ of a function on a Jordan arc, curve or closed region is introduced. After obtaining auxiliary results (Theorems 2.1–2.7) on $\omega(\delta)$, the author deals with the problems α and β. In either case theorems are proved deducing the degree of approximation of the $p_n(z)$ or $\pi_n(z)$ from the continuity properties of $u(z)$ (in some results, of $f(z)$), and converse theorems; and approximation both in the Tchebycheff sense and in integral measure ($\int_C|u(z)|^p ds$ or $\int\int_{\bar{C}}|u(z)|^p dt$, $p>0$) is treated. Examples are given (see Theorems 3.1; 5.3; 9.8). If the curves C are analytic, and $\partial^k u/\partial s^k$, $k=0$ or $1, 2, \cdots$, exists on C, with continuity modulus $\omega(\delta)$, then there are $p_n(z)$'s such that

$$|u(z)-p_n(z)|\leqq Mn^{-k}\omega(n^{-1}) \quad (z \text{ on } \bar{C},\ n\to\infty).$$

Conversely, if C is of a certain type D, $U(z)$ is defined and $|U(z)-p_n(z)| \leqq n^{-k}\Omega(n)$ on $\bar{C}$, then $U(z)$ is a harmonic function $u(z)$ in C, with continuity modulus $\omega(\delta)$ on $\bar{C}$, where (5.2)

$$\omega(\delta) \leqq M\left[\delta \int_a^{a/\delta} \Omega(x)dx + \int_{1/\delta}^{\infty} x^{-1}\Omega(x)dx\right],$$

$0<\delta \leqq 1/a$, $a>0$, $\Omega(x) \geqq 0$, $\Omega(x)$ is monotone for large x and $\int^{\infty} x^{-1}\Omega(x)dx<\infty$. If $k \geqq 1$, then $f^{(k)}(z)$ exists, with a continuity modulus of the type (5.2).

If C is an analytic Jordan curve, $\rho(z)$ is integrable and $\geqq c\ (>0)$ on $\bar{C}$, $p>1$, and

$$\int\int_{\bar{C}} \rho(z)\,|U(z)-p_n(z)|^p dt \leqq n^{-kp-p-1}R^{-np}(\Omega(n))^p,$$

then $\lim_{n\to\infty} p_n(z)=u(z)$ exists on $\bar{C}_R$, $U(z)=u(z)$ almost everywhere on $\bar{C}$, and $f^{(k)}(z)$ and $\partial^k u/\partial s^k$ exist on $\bar{C}_R$ or C_R, respectively, with continuity moduli of the type (5.2).

A number of auxiliary results on polynomials are deduced, all of them of interest in themselves. They are based on theorems due to Szegö, Walsh and Western. *H. Kober.*

Lisovskiĭ, M. A. On polynomials orthogonal on several contours. Mat. Sbornik N.S. **28**(70), 603–620 (1951). (Russian)

The present paper adds to the work of Privaloff, Walsh, Szegö, and Merriman on the title subject. Let $C=C_1$ designate an analytic Jordan curve in the z-plane, $z=g(w)$ be its exterior mapping function, and C_r the images of $|w|=r \geqq 1$. The problem posed and solved is the determination of all sets of polynomials $p_n(z)=a_nz^n+\cdots$, $p_{n+1}(z)$, $\cdots$ which are orthogonal over C_r, $r \geqq 1$, relative to a positive weight $n(z)$, that is, $\int_{C_r} n(z)p_k(z)\overline{p_l(z)}|dz|=0$ for all $r \geqq 1$, $k \neq l$, and k, $l \geqq n$. As is the case in Szegö's problem where orthogonality is required for k, $l \geqq 0$, the author shows that the curves C_r must be either a family of concentric circles or confocal ellipses. The orthogonal polynomials themselves fall into five types of sets, which are given explicitly, and exhibit a greater variety than in Szegö's case.

P. Davis (Cambridge, Mass.).

Noshiro, Kiyoshi. On the singularities of analytic functions with a general domain of existence. Proc. Japan Acad. **22**, no. 8, 233–237 (1946).

Announcement of results whose proofs have appeared elsewhere [Jap. J. Math. **19**, no. 4, 299–327 (1948); these Rev. **11**, 428]. *W. Seidel* (Rochester, N. Y.).

Dugué, Daniel. Sur les valeurs exceptionnelles de Julia et un problème qu'elles soulèvent. C. R. Acad. Sci. Paris **233**, 841–842 (1951).

The author completes his previous results [same C. R. **232**, 380–381 (1951); these Rev. **12**, 601] on Picard exceptional values and states a similar result for Julia exceptional values. He also states that any Julia exceptional value (of a meromorphic function) is an asymptotic value.

S. Agmon (Houston, Tex.).

Swinnerton-Dyer, H. P. F. On a conjecture of Hardy and Littlewood. J. London Math. Soc. **27**, 16–21 (1952).

Let a be an integer greater than 2, and let $0<\lambda<1$. Then the function

$$f(z)=\sum_{n=1}^{\infty} \epsilon_n z^{a^n}/(1-z^{2a^n}) \quad (\epsilon_n=\pm 1)$$

is known to have the property

$$I_\lambda=\int_0^{2\pi} |f(re^{i\theta})|^\lambda d\theta = O(|\log(1-r)|^\lambda);$$

for almost all sequences $\{\epsilon_n\}$ the integral I_λ is unbounded. The author now finds a constant A such that, for each integer a greater than 3, for some function $r(a)$ less than 1 and independent of λ, and for all sequences $\{\epsilon_n\}$, the inequality $I_\lambda>A(a\log a)^{-\lambda}|\log(1-r)|^\lambda$ holds when $r(a)<r<1$.

G. Piranian (Ann Arbor, Mich.).

Shah, S. M. The maximum term of an entire series. VII. Ganita **1**, 82–85 (1950).

[For part VI cf. J. Indian Math. Soc. (N.S.) **14**, 21–28 (1950); these Rev. **12**, 249.] The author constructs an entire function of finite positive order ρ such that (*) $\nu(r)/\log\mu(r)\to\rho$ as $r\to\infty$ while $\nu(r)/r^\rho$ has arbitrarily assigned upper and lower limits. This shows that (*), which is a known consequence of the equivalent relations $\log M(r)\sim Tr^\rho$, $\log\mu(r)\sim Tr^\rho$, $\nu(r)\sim\rho Tr^\rho$, does not imply them. (Here $\mu(r)$ is the maximum term and $\nu(r)$ is the rank of this term.)

In the course of the proof the author shows that in Mercer's theorem [Hardy, Divergent Series, Oxford University Press, 1949, p. 104; these Rev. **11**, 25], that the convergence of $s_n+q(s_0+\cdots+s_n)/(n+1)$ implies that of s_n if $q>-1$, we cannot allow $q=-1$ even if $\{s_n\}$ is bounded and (in a certain sense) slowly oscillating.

R. P. Boas, Jr. (Evanston, Ill.).

Shah, S. M. On exceptional values of entire functions. Compositio Math. **9**, 227–238 (1951).

The author defines a number α as exceptional E for the entire function $f(z)$ if, for some positive nondecreasing $\phi(x)$ with $\int^{\infty}\{x\phi(x)\}^{-1}dx<\infty$,

$$\liminf\,\{\log M(r)\}/\{n(r,\alpha)\phi(r)\}>0;$$

this definition is suggested by the fact that this limit is zero for functions of nonintegral order and for functions of integral order having the same genus as their canonical products [Shah, J. London Math. Soc. **15**, 23–31 (1940); J. Indian Math. Soc. (N.S.) **5**, 179–188 (1942); these Rev. **1**, 307, 400; **4**, 6]. He proves counterparts of known theorems for Borel or Nevanlinna exceptional values. Some of the results are as follows. A Borel exceptional value is exceptional E; a value which is exceptional E is Nevanlinna exceptional and an asymptotic value; the converses of all these statements are false. The property of having α as an E-exceptional value is invariant under a change of origin. Further theorems deal with meromorphic functions and with values exceptional E with respect to simple zeros.

R. P. Boas, Jr. (Evanston, Ill.).

Lapin, G. P. On interpolation in the class of entire functions of finite order and finite type. Mat. Sbornik N.S. **29**(71), 565–580 (1951). (Russian)

The author obtains necessary and sufficient conditions, bearing on $\{\lambda_n\}$ and $\{p_n\}$, for the existence of an entire function $\omega(z)$ of order ρ and finite type such that $\omega^{(i-1)}(\epsilon^q\lambda_n)=a_{niq}$, $n=1, 2, \cdots$; $i=1, 2, \cdots, p_n$; $q=0, 1, \cdots, m-1$, where $\epsilon=e^{-2\pi i/m}$, the a_{niq} being entirely arbitrary. The case where $p_n=1$ for all n was discussed by Leont'ev [Doklady Akad. Nauk SSSR (N.S.) **66**, 153–156 (1949); these Rev. **10**, 695].

R. P. Boas, Jr. (Evanston, Ill.).

Ozaki, Shigeo, and Yosida, Tokunosuke. On some properties of multivalent functions. Sci. Rep. Tokyo Bunrika Daigaku. Sect. A. 4, 137–150 (1949).

Several areal theorems for p-valent functions are obtained of which the following is typical. Let

$$w(z)=\frac{1}{z^k}\cdot\left(\sum_0^\infty a_n z^n\right)^\lambda,\quad a_0=1,$$

be single-valued, regular, $|k|$-valent and have no zeros in $0<|z|<1$, with k integral and λ real. On $|z|=1$ let $m\leqq|w(z)|^{1/\lambda}\leqq M$. Then

$$m^2\leqq\sum_{n=0}^\infty\left(1-\frac{\lambda}{k}n\right)|a_n|^2\leqq M^2.$$

An application is made to obtain the following result for the circle of univalence of $f(z)=z\sum_0^\infty a_n z^n$, $a_0=1$. Let $w(z)=z^k(\sum_0^\infty a_n z^n)^\lambda$ be regular and k-valent in $|z|<1$, $\lambda\geqq k$. Then $f(z)$ is univalent in $|z|<r$ where

$$r^4(1-M^{-2/\lambda})<\frac{\lambda}{k}(1-r^2)^2,\quad M=\lim_{r\to1}\max_{|z|=r}|w(z)|.$$

It is also shown that $w(z)$ takes on every value w of the circle $|w|<\frac{1}{4}$ if $\lambda>0$. *M. S. Robertson.*

Ozaki, Shigeo, Ono, Isao, and Ozawa, Mitsuru. On the function-theoretic identities. I. Sci. Rep. Tokyo Bunrika Daigaku. Sect. A. 4, 157–160 (1951).

Let D be a domain of the complex z-plane, with a boundary Γ_D of Jordan curves. Let $\omega=f(z)$ be single-valued and meromorphic in $\bar{D}$, $n(\omega)$ the number of zeros of ω-$f(z)$ in D, and Γ_ω the image by $\omega=f(z)$ of Γ_D. Using a geometrical principle and Green's formula, the authors obtain the following identity

$$\frac{1}{2i}\int_C\bar{\omega}d\omega+\frac{1}{2i}\int_{(B)}n(\omega)\bar{\omega}d\omega=\iint_B n(\omega)df_\omega,$$

where (B) is the boundary of an arbitrary schlicht domain B of the ω-plane, C is the intersection of Γ_ω with B and df_ω is the area element. Extensions of the Shimizu-Ahlfors identity are obtained [L. Ahlfors, Den syvende skandinaviske matematikerkongress, Oslo, 1929, Brøggers, Oslo, 1930, pp. 84–88; T. Shimizu, Jap. J. Math. 6, 119–171 (1929)] as well as the Cartan identity [H. Cartan, C. R. Acad. Sci. Paris 189, 521–525 (1929)]. *M. S. Robertson.*

Ozaki, Shigeo, Ono, Isao, and Ozawa, Mitsuru. On the function-theoretic identities. II. Sci. Rep. Tokyo Bunrika Daigaku. Sect. A. 4, 161–168 (1951).

Continuing the investigation of the paper reviewed above on the distribution of values of a meromorphic function $\omega=f(z)$ in a connected domain D with boundary Γ the authors show that

$$\left|\frac{2}{\pi}\iint_{|\omega|>1}\frac{n(\omega)df_\omega}{(1+|\omega|^2)^2}-\frac{2}{\pi}\iint_{|\omega|<1}\frac{n(\omega)df_\omega}{(1+|\omega|^2)^2}\right|\leqq\frac{1}{2\pi}\int_\Gamma|d\omega|.$$

An application is made to obtain a function-theoretic identity of D. C. Spencer [Amer. J. Math. 65, 147–160 (1943); these Rev. 4, 137]. If $\omega(z)$ is mean p-valent for $R>R_0$ so that

$$\int_{R_0}^R P(R)RdR\leqq\frac{p}{2}(R^2-R_0^2),$$

and if $(n(0)-p)R^\alpha\geqq(n(\infty)-p)R_0^\alpha$, then

$$\int_{(D)}R^{-\alpha}d\phi\geqq0,\quad \alpha>0,\quad \omega=Re^{i\phi}.$$

M. S. Robertson (New Brunswick, N. J.).

Ono, Isao. On some properties of mean multivalent functions. Sci. Rep. Tokyo Bunrika Daigaku. Sect. A. 4, 169–175 (1951).

Applications of the areal theorem [see the review above] are made to mean one-valent functions. Let

$$\omega(z)=\frac{1}{z}+\sum_0^\infty a_n z^n,\quad 0<|z|<1,$$

be mean one-valent for $|\omega|<R<R_0$, then it is shown that $\omega(z)$ is univalent for $|z|<1/\sqrt{2}$. Let

$$\omega(z)=\frac{1}{z-a}+\sum_0^\infty a_n z^n,\quad |a|<1,$$

be meromorphic in $|z|<1$. Then the necessary and sufficient condition that $\omega(z)$ be mean one-valent in $|\omega|<R<R_0$ for a suitable R_0 is that

$$(1-|a|^2)^{-2}\geqq\sum_1^\infty n|a_n|^2.$$

Let $\omega(z)=\sum_1^\infty a_n z^n$, $a_1=1$, be regular, $|\omega(z)|<M$ for $|z|<1$, and mean one-valent in $|\omega|<M$. Then it is shown that $\omega(z)$ is univalent for $|z|^2<1-(1-M^{-2})^{\frac{1}{2}}$. *M. S. Robertson.*

Tsuji, Masatsugu. On meromorphic functions with essential singularities of logarithmic capacity zero. Tôhoku Math. J. (2) 3, 1–6 (1951).

Let M be a bounded closed set in the z-plane of logarithmic capacity zero, let $w=w(z)$ be a single-valued, meromorphic function in the complement of M, and let every point of M be an essential singularity of $w(z)$. Let ω be a transcendental singularity of the inverse function lying over the point $w=a_0$. Denote by F_ρ that neighborhood of ω on the Riemann surface of the inverse function which lies over the circle $|w-a_0|<\rho$ and denote by Δ the region in the z-plane corresponding to F_ρ. New proofs are given of theorems due to Tsuji [Jap. J. Math. 19, 139–154 (1944); these Rev. 8, 508] and Noshiro [ibid. 19, no. 4, 299–327 (1948); J. Math. Soc. Japan 1, 275–281 (1950); these Rev. 11, 428; 13, 224] which establish the multiplicity with which F_ρ covers points of the circle $|w-a_0|<\rho$ and relate it to some topological properties of Δ. *W. Seidel.*

Pfluger, Albert. Quasikonforme Abbildungen und logarithmische Kapazität. Ann. Inst. Fourier Grenoble 2 (1950), 69–80 (1951).

Let D be a Jordan domain in the z-plane, bounded by smooth curves $\gamma_1,\gamma_2,\cdots,\gamma_n$. Let $\Gamma_0=\bigcup_1^k\gamma_j$ be the "inner" boundary, and let $\Gamma_1=\bigcup_{k+1}^n\gamma_j$ be the "outer" boundary of the hypothetical "annulus" (Γ_0,Γ_1). Among all real functions $u(x,y)$, harmonic in D and satisfying $\int_{\Gamma_0}(\partial u/\partial n)\,ds=2\pi$ there is a $H(x,y)$ that minimizes the Dirichlet integral $I[u]=\iint_D(\text{grad }u)^2dxdy$; $H(x,y)$ must be constant on Γ_0, Γ_1. If m is the minimum of $I[u]$, then $M=m/2\pi$ is called the modulus of the annulus (Γ_0,Γ_1). We remark that $1/M$ is essentially the capacity of (Γ_0,Γ_1).

Now let D be mapped in a quasiconformal manner onto D', by $w=f(z)$, and let the dilation coefficient $Q(z)$ satisfy $|Q(z)|\leqq K(\geqq1)$. If $M'=\text{mod }(\Gamma_0',\Gamma_1')$, then $M/K\leqq M'\leqq M$.

This result reduces to one due to Grötzsch for doubly connected domains.

The author applies the preceding results to the study of the behavior of capacity, the transfinite diameter, distortion under the quasiconformal mapping of D. We cite two examples. Let $D_1 \subset D_2 \subset D_3 \subset \cdots$ be a sequence of domains of the type discussed above, and let $G = \bigcup_{n=1}^{\infty} D_n$. If M_n is the modulus of the annulus associated with D_n, and if $\lim M_n = \infty$, then the boundary of G is said to have zero logarithmic capacity. It follows at once that if G is mapped in a quasiconformal manner on G', and if the boundary of G has zero capacity, then so does the boundary of G'. Again, suppose the disc $|z| < r$ is mapped quasiconformally onto a finite domain G' in the z'-plane, such that $z=0$ is held fixed and such that near $z=0$, $|z'| \sim |z|^{1/K}$. Then the disc $|z'| < \frac{1}{4} r^{1/K}$ contains no frontier point of G'. For $K=1$, this is the Koebe-Bieberbach theorem. *M. Reade.*

Goluzin, G. M. Variational method in conformal mapping. IV. Mat. Sbornik N.S. **29**(71), 455–468 (1951). (Russian)

[For parts I–III, see Mat. Sbornik N.S. **19**(61), 203–236 (1946); **21**(63), 83–117, 119–132 (1947); these Rev. **8**, 325; **9**, 421.] Let $f_k(\zeta)$, regular and univalent in $|\zeta| < 1$, map that region onto a region B_k containing $f_k(0) = a_k$, and suppose $B_k \cap B_j = 0$, $j \neq k$, $j, k = 1, 2, \cdots, n$. The author seeks the maximum of $I = \prod_{k=1}^{n} |f_k'(0)|$, for fixed a_k, $k = 1, 2, \cdots, n$. It is proved that: (a) sets of extremal functions exist; (b) for these sets the boundary of B_k consists of piecewise analytic arcs and $\bigcup \bar{B}_k$ covers the entire complex plane; and (c) the extremal functions satisfy a differential equation of the form $(\zeta f'(\zeta))^2 = R(f(\zeta))$ where R is a rational function. Only for $n=2, 3$, is the maximum value of I obtained, namely for $n=2$, $I \leqq |a_1 - a_2|^2$ and for $n=3$,

$$I \leqq |(a_1-a_2)(a_1-a_3)(a_2-a_3)| \, 64/81\sqrt{3}.$$

Equality occurs for $n=2$ if and only if B_1 and B_2 are half-planes with a common boundary which bisects the line segment joining a_1 and a_2, and this result, first obtained by M. A. Lavrent'ev [Trudy Mat. Inst. Steklov. **5**, 159–245 (1934)], is proved by the author by elementary means. In case $n=3$ and the points a_1, a_2, a_3 form the vertices of an equilateral triangle, equality occurs if and only if the regions B_k are sectors with their vertices at the center of the triangle and their common sides bisecting the sides of the triangle. For arbitrary a_1, a_2, a_3, equality occurs when the regions B_k are appropriate bilinear transforms of the sectors in the simpler case. This result is used to prove that if $F(\zeta) = \zeta + \sum_{n=0}^{\infty} \alpha_n \zeta^{-n}$ is meromorphic and univalent in $|\zeta| > 1$, then

$$|(F(\zeta)-F(\epsilon\zeta))(F(\zeta)-F(\epsilon^2\zeta))(F(\epsilon\zeta)-F(\epsilon^2\zeta))| \geqq 3^{3/2} |\zeta|^3 (1-|\zeta|^{-3})^2,$$

where $\epsilon = e^{2\pi i/3}$, and the bound is sharp as the function $F(\zeta) = \zeta(1+\zeta^{-3})^{2/3}$ shows. *A. W. Goodman.*

Meschkowski, Herbert. Über die konforme Abbildung gewisser Bereiche von unendlich hohem Zusammenhang auf Vollkreisbereiche. I. Math. Ann. **123**, 392–405 (1951).

The author considers the problem of mapping a domain B of countably infinite connectivity on a domain bounded only by complete circles. He obtains two theorems which place weaker restrictions on the boundary than previous results. He defines the modulus of a doubly connected domain G to be d/λ where d is the minimum width of G and λ is the inf of the lengths of all curves in G which go around the inner boundary of G and stay a distance at least $d/2$ from the boundary of G. A boundary component of B is called isolated if it has a positive distance from the other boundary components; otherwise, it is called a limit boundary component. In the author's first mapping theorem, it is assumed that the limit boundary components H_ν consist of only a finite number of points. To each H_ν, $\nu = 1, \cdots, n$, should correspond a system $S_{\nu,\mu}$, $\nu = 1, \cdots, n$, $\mu = 1, 2, \cdots$, of mutually disjoint doubly connected regions lying entirely within B and for μ large enough, lying within any arbitrarily small circle about H_ν. He assumes that $\liminf_{\mu\to\infty} d_{\nu\mu}/\lambda_{\nu\mu}$ is positive. In the second mapping theorem, the author assumes that the boundary components of B are analytic (later only rectifiable) and that there are only a finite number of limit boundary components H_ν. Furthermore, if the isolated boundary components B_ν have length l_ν, distance r_ν from all H_μ, and distance δ_ν from all B_μ, $\mu \neq \nu$, then there should exist two positive constants k_1 and k_2 such that $\delta_\nu > k_1 l_\nu$ and $\delta_\nu > k_2 r_\nu$ for all ν. *G. Springer.*

★Grunsky, H. Über Tschebyscheffsche Probleme. Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 241–246. Amer. Math. Soc., Providence, R. I., 1952.

The author gives a method for treating Tchebycheffian problems in the complex domain which is based on convexity considerations (consequences of the origin's lying in the convex hull of a given set). Application is made to an extremal problem treated by the reviewer [Trans. Amer. Math. Soc. **55**, 349–372 (1944); these Rev. **5**, 259] concerning the Hadamard three circle theorem for functions analytic throughout the interior of the unit circle and to a related more general problem. *M. Heins* (Providence, R. I.).

Goluzin, G. M. On majoration of subordinate analytic functions. II. Mat. Sbornik N.S. **29**(71), 593–602 (1951). (Russian)

[For part I see the same vol., 209–224 (1951); these Rev. **13**, 223.] The author solves several extremal problems for regular univalent functions $f(z)$ and $F(z)$ for which $f(z)$ is subordinate to $F(z)$ in a region R. (1) A new proof for the best value of r_0 such that, when R is the unit circle, $f(0) = F(0) = 0$ and $\arg f'(0) = \arg F'(0) = 0$, then $|f(z)| \leqq |F(z)|$ in $|z| < r_0$ [Biernacki, Mathematica, Cluj **12**, 49–64 (1936); the author, same Sbornik N.S. **6**(48), 383–388 (1939); these Rev. **1**, 308]. (2) Under the hypotheses of (1), the best r_0 such that $|f'(z)| \leqq |F'(z)|$ is $3-2^{3/2}$. (3) The same problem as in (2) when $f(z)$ and $F(z)$ are of the form $z\phi(z^p)$, $p \geqq 1$; here $r_0 = \{(p+1)^{1/2} - p^{1/2}\}^{2/p}$. (4) The problem corresponding to (2) when R is $|z| > 1$, $f(\infty) = F(\infty) = \infty$, $\arg f'(\infty) = \arg F'(\infty)$. *R. P. Boas, Jr.* (Evanston, Ill.).

★Valiron, Georges. Les notions de fonction analytique et de surface de Riemann. Congrès International de Philosophie des Sciences, Paris, 1949, vol. III, Philosophie Mathématique, Mécanique, pp. 27–35. Actualités Sci. Ind., no. 1137. Hermann & Cie., Paris, 1951.

Oka, Kiyoshi. Sur les fonctions analytiques de plusieurs variables. VIII. Lemme fondamental. J. Math. Soc. Japan **3**, 204–214 (1951).

Verf. kündigt mit der vorliegenden Arbeit an, dass er die Reihe seiner fundamentalen Untersuchungen über 1) die Cousinschen Probleme, 2) die Entwicklungssätze und 3) die

verschiedenen lokalen und globalen Konvexitäten in der Funktionentheorie mehrerer Veränderlichen auf nichtschlichte Gebiete, die auch Verzweigungsmannigfaltigkeiten enthalten, ausdehnen wird. Er wendet sich zunächst Problem 2) zu und leitet darüber ein vorbereitendes Lemma ab: "Besitzt ein (reguläres) Ideal $\mathfrak{F}$ mit einem unbestimmten Gebiet im $R^{2(n+s)}$ der Veränderlichen $z_1, \cdots, z_n, y_1, \cdots, y_s$ lokale Pseudobasen, so besitzt unter einer wenig einschränkenden Voraussetzung auch das Ideal I mit unbestimmtem Gebiet, das durch Projektion aus $\mathfrak{F}$ in den Raum R^{2n} der Veränderlichen $z_1, \cdots, z_n$ entsteht, lokale Pseudobasen". Zu den Begriffen reguläres Ideal $\mathfrak{F}$ (hier schlechtweg Ideal $\mathfrak{F}$ genannt) mit unbestimmtem Gebiet sowie lokale Pseudobasis siehe das Referat der siebenten Abhandlung des Verfassers [Bull. Soc. Math. France **78**, 1–27 (1950); diese Rev. **12**, 18]. Der Begriff der Projektion eines Ideals wird in dieser Arbeit zuerst eingeführt. Zu den Übergängen zwischen regulären Funktionen $F(z_1, \cdots, z_n)$ in Polyedergebieten: $f_j \subset \mathfrak{G}_j$, $j=1, \cdots, s$; $|z_k|<1$, $k=1, \cdots, n$ des Raumes der Veränderlichen $z_1, \cdots, z_n$ und regulärer Funktionen des Zylindergebietes $|z_k|<1$, $y_j \subset \mathfrak{G}_j$ des Raumes der $(n+s)$ Veränderlichen $z_1, \cdots, y_s$, die durch die Funktion F auf der Hyperfläche $y_j=f_j(z_1, \cdots, z_n)$ induziert werden, siehe schon vom Verf. Teil I [J. Sci. Hiroshima Univ. Ser. A. **6**, 245–265 (1936)]. Gerade dieser Ansatz ist eine der wesentlichsten Leistungen von Oka.—Über die Verwertung dieser Übergänge, um die Lösung der obigen Probleme von nichtschlichten auf höherdimensionale schlichte Gebiete zurückzuführen, siehe schon H. Behnke und K. Stein [Nachr. Ges. Wiss. Göttingen. Fachgruppe I. **1**, 195–202 (1939)].
H. Behnke (Münster).

Moisil, Gr. C. On a generalization of the idea of monogeneity given by V. S. Fedorov. Acad. Repub. Pop. Române. Bul. Şti. A. **1**, 959–964 (1949). (Romanian. Russian and French summaries)

Let f_k, g_k be real functions of $x_1, x_2, \cdots, x_n$, for $k=1, 2, \cdots, m$, and let $e_1, e_2, \cdots, e_m$ be units in an associative and commutative hypercomplex number system. If $f=\sum_1^m f_k e_k$, $g=\sum_1^m g_k e_k$, then Federoff [Mat. Sbornik N.S. **18**(60), 353–378 (1946); these Rev. **8**, 25] has called f a monogenic function with respect to g if there exists a hypercomplex function h such that $df/dg=h$. As the author shows, this last is equivalent to $\partial f/\partial x^i = h\partial g/\partial x^i$. The last relation is a condition under which the author says f is F-monogenic with respect to g. *M. Reade* (Ann Arbor, Mich.).

Moisil, Gr. C. On monogenic functions in the sense of V. S. Fedorov. Acad. Repub. Pop. Române. Bul. Şti. Ser. Mat. Fiz. Chim. **2**, 545–556 (1950). (Romanian. Russian and French summaries)

The author studies monogenic hypercomplex functions in a commutative algebra, obtaining analogues of the Cauchy integral equation and the Cauchy integral formula; examples are given. *E. F. Beckenbach* (Princeton, N. J.).

Theory of Series

Sierpiński, W. Sur la convergence absolument uniforme des séries de fonctions. Ganita **1**, 97–101 (1950).

The author calls an infinite series of functions "absolutely uniformly convergent" on a set A if it converges uniformly on A whatever the order of its terms may be. [The notation "unconditionally uniformly convergent" would perhaps be preferable.] He shows that $\sum_{k=0}^\infty (1-x)(-x)^k$ on $A=[0, 1]$ is an example of a series which is absolutely and uniformly convergent, but not absolutely uniformly convergent on A. Then the author proves the theorem: In order that the series of functions $\sum_{n=1}^\infty f_n(z)$, defined on a set A of complex numbers and with complex numbers as values, be absolutely uniformly convergent on A, it is necessary and sufficient that the series $\sum_{n=1}^\infty |f_n(z)|$ be uniformly convergent.
A. Rosenthal (Lafayette, Ind.).

Zamansky, Marc. Sur la sommation des séries divergentes. C. R. Acad. Sci. Paris **233**, 908–910 (1951).

Attention is confined to series $\sum u_n$ for which $\lim u_n=0$. Let $g(u)$ be continuous and have bounded variation over $0 \leqq u \leqq 1$, and let $g(0)=1$. A series is evaluable (g) to L if $\lim T_n(g)=L$ where

$$T_n(g)=\sum_{k=1}^n g(k/n)u_k.$$

When p is a nonnegative integer, conditions on $g(u)$ are given which imply that each series of the type considered which is evaluable by the Cesàro method (C, p) is also evaluable (g). In studying a converse problem, the author compares transforms of $\sum u_n$ such as

$$H_n{}^s(g)=\frac{s}{n^s}\sum_{k=1}^n k^{s-1}T_k(g)$$

with transforms obtained by replacing $g(u)$ by functions such as

$$G_s(g, u)=su^s\int_u^1 \frac{g(t)}{t^{s+1}}dt.$$

There are no details. It is indicated that results are obtainable which imply equivalence of the Cesàro, Hölder, and Riesz methods of the same positive integer order p; but the scope of the author's conclusions would seem to be limited by his initial statement that he treats only series $\sum u_n$ for which $\lim u_n=0$. *R. P. Agnew* (Ithaca, N. Y.).

Zamansky, Marc. Sur la sommation des séries divergentes et les théorèmes taubériens. C. R. Acad. Sci. Paris **233**, 999–1001 (1951).

The author asserts and purports to prove the following false proposition. Let $g(u)$ be a polynomial in u having a zero of order $m \geqq 0$ at $u=1$. Let $\sum u_n$ be a series for which $\lim u_n=0$, and let

$$(*) \qquad T_n=\sum_{k=1}^n g(k/n)u_k, \qquad n=1, 2, 3, \cdots.$$

Then $\lim T_n$ exists if and only if $\sum u_n$ is evaluable by the Cesàro method (C, m). It is said that the general case $m>0$ is easily deduced from the simplest case in which $g(1)\neq 0$ and accordingly $m=0$. Transformations used in the paper reviewed above are used here.

That the proposition is false even in case $m=0$ is shown by the simple example in which $g(u)=u$. We put $s_n=u_1+\cdots+u_n$ and write (*) in the form

$$T_n=-\frac{1}{n}\sum_{k=1}^n s_k+\left(1+\frac{1}{n}\right)s_k.$$

In case $\sum u_n$ is the divergent series for which $s_n=\log n$, Stirling's formula enables us to show that $\lim T_n$ exists;

but lim $u_n=0$ and the author's proposition would, if it were valid, imply that lim T_n cannot exist. Consideration of the same function $g(u)=u$ and the series $\sum u_n$ for which $s_n=\log\log(n+2)$ shows that replacement of the Tauberian condition lim $u_n=0$ by the much stronger Tauberian condition lim $nu_n=0$ results in a weaker proposition which is likewise false. *R. P. Agnew* (Ithaca, N. Y.).

Szász, Otto. Tauberian theorems for summability (R_1). Amer. J. Math. **73**, 779–791 (1951).

A series $\sum_1^\infty a_n$ is called summable (R_1) or summable (R, 1) to s if $(2/\pi)\sum s_n n^{-1}\sin nh\to s$ or $\sum a_n(nh)^{-1}\sin nh\to s$, correspondingly, as $h\to 0$. The author has given several theorems of Tauberian type for the method (R, 1) [same J. **64**, 575–591 (1942); **67**, 389–396 (1945); Bull. Amer. Math. Soc. **49**, 885–893 (1943); these Rev. **4**, 37; **7**, 12; **5**, 117]. Now he proves similar theorems for the method (R_1). For example, summability (R_1) is implied by either

$$\text{(a)}\qquad \rho_n=\sum_n^{2n}(|a_\nu|-a_\nu)=O(1)$$

and Abel summability, or (b) $\rho_n=O(n^{1-\delta})$, $\delta>0$ and $s_n-s=o(1/\log n)$, or (c) $\sum_1^n|s_\nu|^{-\alpha}=O(n^{1-\alpha})$, $0<\alpha<1$, and summability $(C, 1-\alpha)$. The Lebesgue constants of the method (R_1) are bounded. *G. G. Lorentz.*

Iyengar, K. S. K. Notes on summability. II. On the relation between summability by Nörlund means of a certain type and summability by Valiron means. Half-Yearly J. Mysore Univ. Sect. B., N.S. **4**, 161–166 (1944).

For each positive integer k, the coefficients $b_{k,n}$ determined by

$$\left[\sum_{j=0}^\infty x^j/(j+1)\right]^k=\sum_{n=0}^\infty b_{k,n}x^n$$

generate a Nörlund transformation

$$\sigma_n=\sum_{r=0}^n b_{k,r}S_{n-r}\Big/\sum_{r=0}^n b_{k,r}$$

by which a sequence S_n is evaluable (N, k) to σ if $\sigma_n\to\sigma$ as $n\to\infty$. This is compared with the Valiron method (V, r) by which the sequence S_n is evaluable to σ if $\sigma_n\to\sigma$ as $n\to\infty$ where

$$\sigma_n=(2\pi n^{2-r})^{-\frac{1}{2}}\sum_{p=0}^\infty S_p\exp\left[-(p-n)^2/2n^{2-r}\right].$$

It is shown that if S_n is evaluable (N, k) to σ for some positive k, then S_n is evaluable (V, r) to σ for each r for which $1\leqq r<2$. *R. P. Agnew* (Ithaca, N. Y.).

Macphail, M. S. Some theorems on absolute summability. Canadian J. Math. **3**, 386–390 (1951).

Mazur [Studia Math. **2**, 40–50 (1930)] and Banach [Théorie des opérations linéaires, Warsaw, 1932, pp. 90–95] have shown that a perfect matrix method of summability A is consistent with every regular method B which is not weaker than A. The author's main result is an analogue of this theorem for the so called l-l methods, namely, series-to-series transformations $y_r=\sum_k a_{rk}x_k$ with the property that $\sum|x_k|<\infty$ implies $\sum|y_r|<\infty$. Applications are given to series $\sum u_k$ with $u_k=O(\rho^k)$, and to the method of Euler-Knopp. *J. D. Hill* (East Lansing, Mich.).

Szász, Otto. On some trigonometric transforms. Pacific J. Math. **1**, 291–304 (1951).

The method of summation of a series $\sum_1^\infty u_\nu$ by the transformations

$$D_n=\sum_{\nu=1}^n u_\nu\left[\frac{\sin \nu t_n}{\nu t_n}\right]^k,\quad k\geqq 1,\ \text{integral},$$

is discussed and the following result obtained: If

$$t_n=p\pi/n+O(n^{-2}),$$

where p is a positive integer, then the above method includes (C, k)-summability. This, and some similar results, are not entirely new [compare Rogosinski, Math. Z. **41**, 75–136 (1936), where a general theory of such methods is given]. *W. W. Rogosinski* (Newcastle-upon-Tyne).

***Karamata, J. Le développement et l'importance de la théorie des séries divergentes dans l'analyse mathématique.** Premier Congrès des Mathématiciens et Physiciens de la R.P.F.Y., 1949. Vol. II, Communications et Exposés Scientifiques, pp. 99–119. Naučna Knjiga, Belgrade, 1951. (Serbo-Croatian. French summary)

Berghuis, J. An approximation connected with cos n and sin n. Computation Dept., Math. Centrum, Amsterdam, Rep. **R 72**, i+9 pp. (1951).

The author obtains results for sin n and cos n, n a positive integer, which are analogous to theorems conjectured by Ramanujan and proved by Szegő and Watson for e^n and conjectured by Aitken and proved by Copson [Proc. Edinburgh Math. Soc. (2) **3**, 201–206 (1933)] for e^{-n}. For instance, the power series for $f(2n)=\cos 2n$ is truncated after the term in $(2n)^{2n-2}$, and an asymptotic series in descending powers of $2n$ is found for the coefficient $f^{(2n)}(\xi)$, $0<\xi<2n$, of the remainder term. This asymptotic series is obtained from an integral form for the remainder, the limits of integration being from 0 to $2n$. Expansion of the integrand and approximation of the resulting terms, by extension of the upper limit to infinity, leads to integrals of the form $\int_0^\infty e^{-x}x^n\sin x\,dx$. Tables of these integrals for $n=0, 1, \cdots, 30$ are appended. *P. W. Ketchum* (Urbana, Ill.).

Doetsch, Gustav. Beitrag zur Asymptotik der durch komplexe Integrale dargestellten Funktionen. Ann. Scuola Norm. Super. Pisa (3) **5**, 105–119 (1951).

Let B be the region bounded on the right by a vertical straight line and on the left by the rays emanating from s_0 at the angles $\pm\psi$ where $\frac{1}{2}\pi<\psi<\pi$. Let $f(s)$ be analytic in B and continuous in its closure except at s_0. We define

$$F(t)=\frac{1}{2\pi i}\int_{W_0}f(s)e^{ts}ds$$

where W_0 is the path consisting of the lines $|\arg(s-s_0)|=\psi$, $|s-s_0|\geqq\delta$, and of the circular arc

$$|s-s_0|=\delta,\quad |\arg(s-s_0)|\leqq\psi.$$

If $f(s)\sim\sum_{n=0}^\infty c_n(s-s_0)^{\lambda_n}$ as $s\to s_0$ in B where $\lambda_0<\lambda_1<\cdots$ (finitely many λ_n's may be negative), and if $f(s)=O(e^{k|s|})$ as $|s|\to\infty$ on W_0, then

$$F(t)\sim\sum_{n=0}^\infty c_n[\Gamma(-\lambda_n)]^{-1}t^{-\lambda_n-1}\quad\text{as}\quad t\to+\infty.$$

Other formulas of this type are established; the author remarks that many familiar asymptotic expansions may be obtained as special cases. *I. I. Hirschman.*

Brun, Viggo. Wallis's and Brouncker's formulas for π. Norsk Mat. Tidsskr. **33**, 73–81 (1951). (Norwegian)

The author gives a discussion of the historical background of Wallis' formula

$$\frac{\pi}{4}=\frac{2\cdot4\cdot4\cdot6\cdot6\cdot8\cdots}{3\cdot3\cdot5\cdot5\cdot7\cdot7\cdots}.$$

An account is given of Wallis' discovery of this result by boldly setting $n=\frac{1}{2}$ in a formula equivalent to

$$\binom{2n}{n}(2n+1)\int_0^1(x-x^2)^n dx=1$$

and of Brouncker's criticism of the proof and an alternative derivation via continued fractions. The author gives a pre-Newtonian proof of Gregory's series for $\pi/4$ which Brouncker or Wallis could have derived. *D. H. Lehmer.*

Fourier Series and Generalizations, Integral Transforms

✓***Hardy, G. H., and Rogosinski, W. W. Fourier series.** 2nd ed. Cambridge Tracts in Mathematics and Mathematical Physics, no. 38. Cambridge, At the University Press, 1950. x+100 pp. 10s. 6d.

This edition is essentially the same as the first [1944; these Rev. **5**, 261] with known misprints and mistakes corrected.

Timan, A. F. A generalization of some results of A. N. Kolmogorov and S. M. Nikol'skiĭ. Doklady Akad. Nauk SSSR (N.S.) **81**, 509–511 (1951). (Russian)

Given any function $f\varepsilon L$ of period 2π, and any number $0\leqq\theta_n\leqq1$, let

$$\sigma_n(f,\theta_n;x)=\int_0^\infty\left\{f\left(x+\frac{t}{n\theta_n}\right)+f\left(\theta-\frac{t}{n\theta_n}\right)\right\}G_n(t)dt,$$

where

$$G_n(t)=\frac{1}{\pi t^2}\left[\cos\frac{1-\theta_n}{\theta_n}t-\cos\frac{t}{\theta_n}\right].$$

The author calls the σ_n the operators of Akhieyser and Levitan (who considered them also in the case of f non-periodic and such that $f(x)(1+x^2)^{-1}$ is integrable over $(-\infty,+\infty)$; for periodic f the σ_n had been introduced by de la Vallée Poussin [see, e.g., Leçons sur l'approximation . . . , Gauthier-Villars, Paris, 1919, p. 23]. For $\theta_n=1$ the σ_n are the $(C,1)$-means of the Fourier series of f, and for the $\theta_n=1/n$ they coincide with the partial sums of that series. Let $KW^{(r)}H^{(\alpha)}$, where $r\geqq0$, $0\leqq\alpha\leqq1$, denote the class of functions $f(x)$ of period 2π such that $f^{(r)}(x)$ exists and satisfies a Lipschitz condition of order α with coefficients K (here r may also be fractional). Let

$$E_{\sigma_n}(KW^{(r)}H^{(\alpha)};x)=\sup_{f\varepsilon KW^{(r)}H^{(\alpha)}}|f(x)-\sigma_n(f,\theta;x)|.$$

The author states without proof the formula

$$E_{\sigma_n}(KW^{(r)}H^{(\alpha)};x)$$
$$=2^{\alpha+1}\pi^{-2}n^{-r-\alpha}K\log(1/\theta_n^*)\int_0^{\pi/2}t^\alpha\sin t\,dt+O(n^{-r-\alpha}),$$

where $\theta_n^*=\max\{\theta_n,1/n\}$ and $O(1)$ is a quantity uniformly bounded for all n and all $\{\theta_n\}$ such that $\theta_n\leqq1-\theta(\theta>0)$. *A. Zygmund* (Chicago, Ill.).

Postnikov, A. On some trigonometric inequalities. Doklady Akad. Nauk SSSR (N.S.) **81**, 501–504 (1951). (Russian)

Let $S=\sum_{x=1}^N e^{2\pi i f(x)}$, where $f(x)$ is real. It is well known that, if $\Delta f(x)=f(x+1)-f(x)$ is monotonic and

$$0<\theta<\Delta f(x)<1-\theta,$$

then $|S|\leqq1/\theta$. This paper develops a number of extensions with the monotonic condition relaxed. Thus (Theorem 3), if $\Delta f(1),\cdots,\Delta f(N-1)$ contains a monotonic subsequence of length l, and $\epsilon=\max|\Delta f(x)-\Delta f(y)|$, then

$$|S|\leqq\{1+\pi\epsilon(N-l-1)\}/\theta;$$

and (Theorem 4), if $\varphi(x)$ is monotonic, $0<\varphi(x)<1-\theta$, and $|\Delta f(x)-\varphi(x)|<\epsilon$, then $|S|\leqq\{1+\pi\epsilon(N-1)\}/\theta$. The proofs are by an adaptation of the analytical counterpart of Kuz'min's geometrical proof of the original inequality. There are numerous misprints. *A. E. Ingham.*

Grison, Emmanuel. De l'usage des inégalités de Harker-Kasper. Acta Cryst. **4**, 489–490 (1951).

The author discusses relations among the inequalities for Fourier coefficients of positive functions given by Harker and Kasper and by Karle and Hauptman [Acta Cryst. **1**, 70–75 (1948); **3**, 181–187 (1950); these Rev. **12**, 496] and shows that the inequality which in the one-dimensional case would be

$$(c_n\pm c_m)^2\leqq(1\pm c_{n+m})(1\pm c_{n-m}),\qquad c_0=1,$$

for an even function with complex Fourier coefficients c_n is more powerful than the others. *R. P. Boas, Jr.*

Ossicini, Alessandro. Sulla sommabilità delle serie di Legendre. Boll. Un. Mat. Ital. (3) **6**, 218–225 (1951).

Suppose $f(x)(1-x^2)^{-1/4}$ is Lebesgue integrable in the interval $-1\leqq x\leqq1$ and $s_n(x)$ is the nth partial sum of the Legendre series of $f(x)$. For k a positive integer,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{\nu=1}^n[s_\nu k(x)-f(x)]=0$$

for almost all x in $-1<x<1$. The proof requires x to be in the Lebesgue set, although the theorem as stated allows a larger set. *P. Civin* (Eugene, Ore.).

Campbell, Robert. Sur la sommabilité et la dérivabilité de la série de Weber d'une fonction. C. R. Acad. Sci. Paris **233**, 910–912 (1951).

The author continues his investigations of the summability of Hermite series [same C. R. **231**, 1024–1026 (1950); **233**, 596–598 (1951); these Rev. **12**, 406; **13**, 341] and briefly indicates the application of the summability theory to the differentiation of such series. *A. Erdélyi.*

Widder, D. V. Necessary and sufficient conditions for the representation of a function by a Weierstrass transform. Trans. Amer. Math. Soc. **71**, 430–439 (1951).

The author proves that a set of necessary and sufficient conditions for $f(x)$ to be representable for all x as a Weierstrass-Stieltjes (=WS) transform

$$(1)\qquad f(x)=(4\pi)^{-1}\int_{-\infty}^{\infty}\exp[-\tfrac{1}{4}(x-u)^2]d\alpha(u)$$

with a non-decreasing $\alpha(u)$ is that (1) $f(x)$ be entire, (2) $\exp(-\frac{1}{4}y^2)|f(x+iy)|\leqq M(R)$ for every $R>0$, and (3) $\exp(-tD^2)[f]\geqq0$ for $0<t<1$, $-\infty<x<\infty$. Two interpre-

tations are given for the differential operator $\exp(-tD^2)$, namely

$$(4\pi t)^{-\frac{1}{2}}\int_{-\infty}^{\infty}\exp[-y^2/(4t)]\cos(yD)[f]dy$$

with

$$\cos(yD)[f]=\sum_{k=0}^{\infty}\frac{(-y^2)^k}{(2k)!}f^{(2k)}(x),$$

and

$$(2)\quad (4\pi t)^{-\frac{1}{2}}\int_{-\infty}^{\infty}\exp[-y^2/(4t)]f(x+iy)dy\equiv v(x,1-t).$$

These interpretations are equivalent for the class of functions under consideration. The necessity of (1) and (2) is immediate, (3) follows upon substituting (1) into (2) and reducing. The sufficiency is based upon the fact that $v(x,t)$ is a non-negative solution of the heat equation tending to $f(x)$ as $t\to 1-0$. By an earlier result of the author [Trans. Amer. Math. Soc. **55**, 85–95 (1944); these Rev. **5**, 203], $v(x,t)$ is then a WS-transform of a non-decreasing function $\alpha(u)$ in the strip $0<t<1$ and the integral may be shown to converge for $t=1$ and represent $f(x)$. Other necessary and sufficient conditions for representability of $f(x)$ in terms of W- or WS-transforms have been found by A. Gonzáles Domínguez [Ciencia y Técnica **42**, 283–331 (1941); these Rev. **12**, 330] and H. Pollard [Duke Math. J. **10**, 59–65 (1943); these Rev. **5**, 178, 328; **6**, 334]. (It might be observed that if $f(x)=W[g]$, then the reviewer has shown [Ann. of Math. (2) **27**, 427–464 (1926), esp. part 5] under rather general assumptions that $v(x,1-t)\to g(x)$ almost everywhere as $t\to 1-0$.) *E. Hille* (New Haven, Conn.).

Raymond, François-Henri. Transformées de Hilbert et relations de Bayard-Bode. Ann. Télécommun. **6**, 262–272 (1951).

Let $f(z)$ be a function which is analytic in the half-plane $\Re z\geqq 0$. The Hilbert transforms, which relate the real and imaginary parts of f on the imaginary axis, are found by a direct use of Cauchy's integral formula. In a well known network application f is the logarithm of a rational function. The modification of Hilbert's formulae when the rational function has zeros in the half-plane is found and the physical implications are discussed. *R. J. Duffin.*

Bloch, Pierre Henri. Ueber eine Laplace-Transformierte, welche in keiner Halbebene beschränkt ist. Compositio Math. **9**, 289–292 (1951).

The author constructs a function $F(t)$ such that its Laplace transform $f(s)=\int_0^\infty e^{-st}F(t)dt$ is convergent for all complex s and such that $f(s)$ is bounded in no half plane. This answers a question raised by Doetsch [Handbuch der Laplace-Transformation, Bd. I, Birkhäuser, Basel, 1950; these Rev. **13**, 230]. *I. I. Hirschman, Jr.*

Barrucand, Pierre A. La transformation de Mellin et ses applications. Ann. Télécommun. **5**, 381–388 (1950).

Expository paper.

Bose, S. K. A note on Whittaker transform. Ganita **1**, 16–22 (1950).

Other papers of the author on the same subject have appeared earlier [Bull. Calcutta Math. Soc. **41**, 9–27, 59–67, 68–76, 221–222 (1949); **42**, 43–48 (1950); J. Indian Math. Soc. (N.S.) **14**, 29–34 (1950); these Rev. **11**, 28, 173, 174; **12**, 95, 256]. Four of the five theorems of the present paper are concerned with the degenerate case of Whittaker's functions, and are obvious deductions from known features of this case (Theorem I, $M_{k,m}$ and $W_{k,m}$ are numerical multiples, Theorem II, explicit formula for $W_{k,m}$, Theorem IV, recurrence relation, Theorem V, differentiation formula). Theorem III gives the result of a "Whittaker transformation" followed by a Laplace transformation, and was suggested by S. C. Mitra. There are examples illustrating the theorems. *A. Erdélyi* (Pasadena, Calif.).

Fränz, Kurt. Über Signale gegebener Dauer und kleinster spektraler Breite. Arch. Elektr. Übertragung **5**, 515–516 (1951).

Colombo, Serge. L'utilisation du calcul symbolique dans la recherche mathématique. Ann. Télécommun. **5**, 347–364 (1950).

Expository paper.

Zadeh, Lotfi A. Time-dependent Heaviside operators. J. Math. Physics **30**, 73–78 (1951).

"The present paper is devoted to a brief and formal discussion of some of the basic properties of so-called time-dependent Heaviside operators." If the response of a linear system at time t to a Dirac δ-function applied at time $t-\tau$ is denoted by $W(t,\tau)$, then the familiar formula for the response to a function $u(t)$ is given by $V(t)=\int_0^\infty W(t,\tau)u(t-\tau)d\tau$. Taking Laplace transforms, $H(s;t)$ is defined by $\int_0^\infty W(t,\tau)e^{-s\tau}d\tau$. If the Laplace transform of $u(t)$ is $U(s)$, then obviously

$$v(t)=(2\pi j)^{-1}\int_{c-j\infty}^{c+j\infty}H(s;t)U(s)e^{st}ds.$$

The case where $H(s;t)$ is a random function of t is given special attention. *N. Levinson* (Cambridge, Mass.).

Carstoiu, Ion. Une formule générale opératoire dans le calcul symbolique. C. R. Acad. Sci. Paris **233**, 721–723 (1951).

The general formula is well known, at least since 1935 when it was published, more or less simultaneously, by A. M. Efross [Mat. Sbornik **42**, 699–706 (1935)] and (in a slightly specialized form) by J. P. Schouten [Physica **2**, 75–80 (1935)]. The examples given in the present note are also known. A less well-known formula is stated (without conditions of validity) in the last few lines. If $f(t)\supset\varphi(p)$ and $h(t)[f_1(t)]^u\supset\xi(p,u)$, then

$$h(t)f[\log f_1(t)]\supset(2\pi i)^{-1}\int_{Br_1}\xi(p,u)\varphi(u)u^{-1}du.$$

A. Erdélyi (Pasadena, Calif.).

Harmonic Functions, Potential Theory

Brelot, Marcel. Remarques sur la variation des fonctions sousharmoniques et les masses associées. Application. Ann. Inst. Fourier Grenoble **2** (1950), 101–112 (1951).

Let $\mathcal{E}^n$, $n\geqq 2$, denote the compact space obtained from E^n by the adjunction of a point at infinity, let Ω be a domain, in $\mathcal{E}^n$, and let U denote a family of real functions u such that (i) each u is subharmonic in Ω, (ii) there exists a non-polar set $B\subset\Omega$ and a fixed constant k such that for each u, $u\leqq k$ on B, (iii) there exists a fixed compact set $A\subset\Omega$ and a fixed subharmonic function f, such that for each u, $f\leqq u$ on $\Omega-A$. Then a necessary and sufficient condition that the

family U be uniformly bounded above on each compact subset of Ω is that the corresponding negative masses be uniformly bounded on each compact subset of Ω. This result is used to further develop the idea of "action from a distance", introduced earlier [Bull. Soc. Math. France **73**, 55–70 (1945); C. R. Acad. Sci. Paris **226**, 1499–1500 (1948); these Rev. **7**, 205; **9**, 508]. *M. Reade.*

Pfluger, Albert. À propos d'un mémoire récent de M. Brelot. Ann. Inst. Fourier Grenoble **2** (1950), 81–82 (1951).

Let $u(z)\equiv u(x, y)$ be subharmonic for $|z|<\infty$ and let $m(r, u^+)\equiv(1/2\pi)\int_0^{2\pi}u^+(re^{i\varphi})d\varphi$. Then

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{u^+(Re^{i\varphi})(R^2-r^2)}{R^2+r^2-2rR\cos(\theta-\varphi)}d\varphi$$

is a harmonic majorant for u in the circle $|z|<R$. Hence it follows that if $m(r, u^+)=O(r^k)$ $[o(r^k)]$ as $r\rightarrow\infty$, then $u^+=O(r^k)$ $[o(r^k)]$ as $r\rightarrow\infty$. If u is subharmonic in a domain containing $0<a\leqq|z|<\infty$ then set $M=\max_{a\leqq z\leqq a+1}u^+$. Then the function v, defined to be 0 for $|z|\leqq a$, and $(u-M)^+$ for $|z|>a$, is subharmonic for $|z|<\infty$. Hence, v lends itself to the first result above. Similar results can be obtained in n dimensions. The preceding results are slight extensions, with simple proofs, of results due to Brelot [same Ann. **1**, 121–156 (1950); these Rev. **12**, 258]. *M. Reade.*

Lax, Peter D. A remark on the method of orthogonal projections. Comm. Pure Appl. Math. **4**, 457–464 (1951).

The author gives in an elegant formulation an extension of an existence proof of the reviewer and M. Schiffer [Ann. of Math. **52**, 164–187 (1950); these Rev. **12**, 89] which yields the solution of the Dirichlet problem for Laplace's equation in the plane. Let f be a sufficiently differentiable function in a region D with sufficiently smooth boundary. It is easily shown that there exists a harmonic function u in D for which the Dirichlet integral $\iint_D(v_x^2+v_y^2)dxdy$ of the difference $v=f-u$ is a minimum. By considering the scalar product of v and the elementary solution $\log|z-w|$ of Laplace's equation, the author is able to prove directly that u has the same boundary values as f. This technique is discussed in its relation to Dirichlet's principle and orthogonal projection. *P. R. Garabedian.*

Nicolescu, Miron. Direct solution of a boundary problem for biharmonic functions defined in a hyperspherical domain. Acad. Repub. Pop. Române. Bul. Ști. Ser. Mat. Fiz. Chim. **2**, 453–459 (1950). (Romanian. Russian and French summaries)

The author studies the problem of finding biharmonic $u(P)$ inside a hypersphere $S_R(0)$ in n dimensions when the first two inner normal derivatives of u on $S_R(0)$ are given. He finds that if $\partial u/\partial n_i\equiv\varphi(P)$, $\partial^2u/\partial n_i^2\equiv\psi(P)$ on $S_R(0)$, then $\int_{S_{R(0)}}[\varphi(P)+R\psi(P)]d\sigma_P=0$. If this last condition holds, then u is given by an integral bearing a strong resemblance to the corresponding solution for the Neumann problem. *M. Reade* (Ann Arbor, Mich.).

Deny, Jacques. Sur la définition de l'énergie en théorie du potentiel. Ann. Inst. Fourier Grenoble **2** (1950), 83–99 (1951).

The author considers three definitions of the energy of a distribution of positive mass μ in R^n, $n\geqq 2$, which he has discussed earlier [Acta Math. **82**, 107–183 (1950); these Rev. **12**, 98]. (A) If $K(x)$ is the usual Newtonian kernel, then the energy of μ is $I_A(\mu)=\int K(x-y)d\mu(x)d\mu(y)$. (B) If K is a positive, symmetric measure, then μ is said to be of finite energy if and only if the measure $K*\mu*\check{\mu}$ exists and has a continuous density. The energy of μ is denoted by $I_B(\mu)$. (C) If the Fourier transform $\mathcal{K}$ of K is a positive function, summable on each compact set, and if $\mathcal{M}$ is the Fourier transform of μ, then the energy of μ is $I_C(\mu)=\int\mathcal{K}|\mathcal{M}|^2dx$, whenever the integral is finite.

The author's main result is the following one. Let K be a measure whose Fourier transform $\mathcal{K}$ is a function for which there exists $p>0$ such that

$$\int\mathcal{K}(1+|x|^2)^{-p}dx<\infty,\quad\int\mathcal{K}^{-1}(1+|x|^2)^{-p}dx<\infty.$$

Let $\mathcal{E}(B)$ ($\mathcal{E}(C)$) denote the set of all positive measures μ for which $I_B(\mu)$ ($I_C(\mu)$) is finite. Then $\mathcal{E}(B)\subset\mathcal{E}(C)$; moreover, under the usual energy-norm, $\mathcal{E}(B)$ is complete in $\mathcal{E}(C)$. The author also gives conditions under which $I_A(\mu)=I_B(\mu)$; these conditions are less restrictive than earlier ones [H. Cartan and J. Deny, Acta Sci. Math. Szeged **12**, 81–100 (1950); these Rev. **12**, 257]. *M. Reade.*

Keller, Heinrich. Über das Anwachsen von Potentialfunktionen im dreidimensionalen Raum. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **83**, 37 pp. (1950).

This paper is concerned with the extension of the Phragmén-Lindelöf theorem and its modifications and refinements (R. and F. Nevanlinna, Carleman, Ahlfors, M. Heins) to harmonic functions of three variables. The first part of the paper is concerned with the behavior of a harmonic function whose domain is a half-space and which satisfies the Phragmén-Lindelöf boundary conditions. Exact extensions of known two-dimensional theorems are given with hemispheres taking over the role of semicircles. The second part of the paper treats the Carleman inequality for harmonic functions in an unbounded region G. It is assumed that $G\cap[z=c]$ is bounded for each c. The argument of Carleman admits extension to this case thanks to an extremal property of the least characteristic value of $u_{xx}+u_{yy}+\lambda^2u=0$ with boundary condition: $u=0$ [H. A. Schwarz, Gesammelte mathematische Abhandlungen, Bd. 1, Springer, Berlin, 1890, pp. 241–269]. An asymptotic study is made for the extended Carleman inequality in the cases of special regions possessing simple symmetry properties. Estimates are obtained in more general cases with the aid of the Rayleigh-Faber theorem. *M. Heins* (Providence, R. I.).

Bucerius, H. Zu Dirichlet's Ableitung des Ellipsoidpotentials. Astr. Nachr. **279**, 238–240 (1951).

By taking as discontinuous factor the Fourier transform of $x^{-1}\sin x$, where the path of integration is the entire axis of reals with an indentation at the origin, the author shows that the analytical difficulties (arising from convergence considerations at the origin) encountered in Dirichlet's method for obtaining the potential of a homogeneous ellipsoid [G. Lejeune-Dirichlet, Werke, vol. I, Reimer, Berlin, 1889, p. 404] may be avoided. *R. G. Langebartel.*

van den Dungen, F. H. Note on the Hamel-Synge theorem. Quart. Appl. Math. **9**, 203–204 (1951).

The theorem of the paper is this: the velocity $\mathbf{v}$ of a compressible fluid, moving in a region V bounded by a closed surface B, vanishes on B if and only if $I=0$, where $I=\int_V(\mathbf{A}\cdot\operatorname{curl}\mathbf{v}-f\operatorname{div}\mathbf{v})dV$, f being an arbitrary harmonic function and $\mathbf{A}$ a conjugate harmonic vector such that (*) $\operatorname{grad}f+\operatorname{curl}\mathbf{A}=0$. For proof, the author shows directly that, on account of (*), $I=-\int_B[\mathbf{A}\cdot(\mathbf{v}\times\mathbf{n})+f\mathbf{v}\cdot\mathbf{n}]dB$, where

n is the unit normal to B. Thus, $\mathbf{v}=0$ on B implies $I=0$ (necessity of condition). The author states that, on account of the arbitrariness of the harmonic f, $I=0$ implies $\mathbf{v}=0$ on B (sufficiency of condition); this is not clear to the reviewer. The condition $I=0$ generalises to three dimensions the condition given for plane incompressible motion by G. Hamel [Göttingen Nachr. Math.-Phys. Kl. **1911**, 261–270] and with better proof by J. Kampé de Feriet [Math. Mag. **21**, 74–79 (1947); these Rev. **9**, 433], and extended to plane compressible motion by the reviewer [Quart. Appl. Math. **8**, 107–108 (1950); these Rev. **12**, 59]. *J. L. Synge.*

Aržanyh, I. S. A new solution of the problem of the computation of a vector from its curl and divergence. Doklady Akad. Nauk SSSR (N.S.) **79**, 29–32 (1951). (Russian)

The curl and the divergence of the unknown vector are specified in a region, and also its normal component on the boundary, which may be interior or exterior. If the given curl-function has a certain special form, the vector can be found by the theory of Neumann's problem. To express the curl-function in this form the theory of Dirichlet's problem is used. A solution of a more general problem is indicated. Hydromechanical and electromagnetic examples are discussed. *F. V. Atkinson* (Ibadan).

Herriot, John G. The polarization of a lens. Pacific J. Math. **1**, 369–397 (1951).

If a conducting solid is introduced into a uniform electric field with direction determined by the unit vector $\mathbf{h}$, the field is changed by the addition of a field of potential ψ, where ψ is harmonic outside the body, behaves like a dipole at infinity and satisfies on the surface of the body the boundary condition $\psi=\mathbf{h}\cdot\mathbf{r}+$constant. The energy $P=\int|\operatorname{grad}\psi|^2d\tau$, where the integral is extended over the exterior of the body, is a quadratic form $P=\sum P_{ik}h_ih_k$ in the components h_i of $\mathbf{h}$. The quantity P is called the polarization of the body in the direction $\mathbf{h}$, and $P_m=P_{1,1}+P_{2,2}+P_{3,3}$, which is independent of coordinates, is called the average polarization. For a spherical body of volume V, $P_m=2V$, and it has been conjectured that for other bodies $P_m>2V$. If C is the electrostatic capacity of the body, $V\leqq\frac{4}{3}\pi C^3$; hence the inequality (*) $P_m+V\geqq 4\pi C^3$ is stronger than $P_m\geqq 2V$. In the present paper, the author proves (*) for several cases of the lens, a lens being a body bounded by two spherical caps. Special or limiting cases of lenses are the spherical bowl (the two caps coincide), the lens with orthogonal caps, two tangent spheres (the two caps reduce to zero), and the symmetric lens (two equal caps). These are the cases for which (*) is proved. The proofs are based on an analysis of explicit representations of P_m and C for a lens in terms of definite integrals given by Schiffer and Szegö [Trans. Amer. Math. Soc. **67**, 130–205 (1949); these Rev. **11**, 515] and Szegö [Bull. Amer. Math. Soc. **51**, 325–350 (1945); these Rev. **6**, 227]. The analysis is very detailed and intricate and it would be unfeasable to attempt to sketch it here.
J. W. Green (Princeton, N. J.).

Differential Equations

★Èl'sgol'c, L. È. Obyknovennye differencial'nye uravneniya. [Ordinary Differential Equations]. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950. 220 pp.

This book is intended for the use of engineers and students of technical schools. Chapters I–III contain standard theorems and methods on ordinary differential equations and systems. In chapter IV special subjects are discussed such as stability, dependence on parameters in the singular case, periodic solutions, equations with retarded arguments and integration machines. The book is written in a clear and rigorous style, but theorems requiring a lengthy and difficult proof are merely stated (with the exception of Picard's theorem whose proof is given in an appendix); there is a list of references where these missing proofs and further developments may be found. The general methods are illustrated by means of many worked-out examples and more than a hundred exercises with solutions.
J. L. Massera (Montevideo).

Demidovič, B. P. On stability in the sense of Lyapunov of a linear system of ordinary differential equations. Mat. Sbornik N.S. **28(70)**, 659–684 (1951). (Russian)

The author considers the problem of determining the asymptotic behavior of the solutions of the linear differential equation $dy/dt=(A+B(t))y$ under certain assumptions concerning the constant matrix A and the variable matrix $B(t)$. He uses the method, first introduced by Cesari [Ann. Scuola Norm. Super. Pisa (2) **9**, 163–186 (1940); these Rev. **3**, 41] of considering the characteristic roots not only of A but also of $A+B(t)$. The results are too detailed to quote.
R. Bellman (Stanford University, Calif.).

Gradštein, I. S. Application of A. M. Lyapunov's stability theory to the theory of differential equations with small coefficients of the derivatives. Doklady Akad. Nauk SSSR (N.S.) **81**, 985–986 (1951). (Russian)

Consider the system

$$(1)\qquad dY_i/dt=h_i(Y,p),\quad i=1,2,\cdots,n,$$

where Y is an n-dimensional vector, and p an l-dimensional parameter-vector. Let S be the set of singular points y^*, p^* of (1) and R a subset of S such that the solution of (1) possesses the following property: For any ϵ, α, $0<\alpha\leqq\epsilon$, there is a $\rho(\epsilon)$, $0<\rho(\epsilon)\leqq\epsilon$, and a $\theta(\rho,\alpha)>0$ such that, for any system of the family (1) which has a singular point entering R, from

$$(2)\qquad |Y_i(0,p)-y_i^*|<\rho,\quad i=1,2,\cdots,n,$$

and $\tau\geqq 0$ there follows

$$|Y_i(\tau,p)-y_i^*|<\epsilon,\quad i=1,2,\cdots,n,$$

and from (2) and $\tau\geqq\theta$ there follows

$$|Y_i(\tau,p)-y_i^*|<\alpha,\quad i=1,2,\cdots,n.$$

Under these circumstances the family of relations (1) is said to define a motion which is uniformly asymptotically stable (u.a.s.) relative to the set R. Theorem: The proposition of an earlier paper [same Doklady **66**, 789–792 (1949); these Rev. **10**, 709] still holds provided that the set R is determined by the condition that the motion defined by the system

$$dy_i/dt=h_i(x,y,t),\quad i=1,2,\cdots,n,$$

where x, t are parameters, is u.a.s. relative to R, and that the Jacobian $|\partial h_i/\partial y_j|\neq 0$ on R.

One obtains criteria for u.a.s. quite naturally from those for asymptotic stability of Lyapunov. A criterion founded on Lyapunov's first method was given by the author explicitly in the paper already cited, and also in an earlier one [ibid. **65**, 789–792 (1949); these Rev. **10**, 708]. In the present paper he produces also a criterion based upon the second method of Lyapunov. *S. Lefschetz.*

Šestakov, A. A. Some theorems on stability in Lyapunov's sense. Doklady Akad. Nauk SSSR (N.S.) 79, 25–28 (1951). (Russian)

Let $\dot{x}_s = X_s^{(m)}(x_1, \cdots, x_n) + L_s(x_1, \cdots, x_n)$, where the $X_s^{(m)}$ are relatively prime homogeneous polynomials of degree m and $L_s = O(r^{m+1})$, $r^2 = x_1^2 + \cdots + x_n^2$. Any solution $(a_1, \cdots, a_n)$ of the system $x_1/X_1^{(m)} = \cdots = x_n/X_n^{(m)}$ is a "critical direction" and, if m is odd, it is called "positive" whenever the ratio $x_1/X_1^{(m)}$ is positive. The following results are proved: (1) If a critical direction exists (m even) or if a positive critical direction exists (m odd), the origin is unstable; (2) if n is odd, there is always a critical direction; (3) as a corollary, the case n odd, m even, is always unstable. Another criterion of instability is derived for the case $m=1$ when several characteristic exponents vanish. *J. L. Massera.*

Šestakov, A. A. On the behavior of the integral curves of a system of n differential equations ($n \geq 3$) near to a singular point of higher order. Doklady Akad. Nauk SSSR (N.S.) 79, 205–208 (1951). (Russian)

Let $\dot{x}_s = X_s(x_1, \cdots, x_n)$, where the X_s vanish at the origin and are holomorphic in its neighborhood; the developments may begin with terms of degree higher than one. The author proves the existence of families of integral curves given by equations $x_i = (a_i + z_{i-1})t^{p_i}$ where the p_i are integers, a_i constants and $z_i \to 0$ as $t \to 0$; the precise statement is too lengthy to be formulated here. There are several printing errors. *J. L. Massera* (Montevideo).

Cashwell, E. D. The asymptotic solutions of an ordinary differential equation in which the coefficient of the parameter is singular. Pacific J. Math. 1, 337–352 (1951).

The differential equation under consideration is

$$w''(s) - [\lambda^2\sigma(s) + \tau(\lambda, s)]w(s) = 0$$

where λ is a large complex parameter and the coefficients are analytic functions. At $s = s_0$ the function $\sigma(s)$ is supposed to have a pole of the second order, and $\tau(\lambda, s)$ has there at most a pole of second order, with principal part independent of λ. The author derives asymptotic expressions, valid in a neighborhood of the point s_0, for a fundamental system of the differential equation. These asymptotic expressions are elementary functions, in contrast to the formulas obtained by R. E. Langer [Trans. Amer. Math. Soc. 37, 397–416 (1935)] in the case that $\sigma(s) = O[(s-s_0)^\nu]$, $\nu > -2$. For the validity of the asymptotic relations, λ must be restricted to a certain sub-domain of the λ-plane. There are four such domains, and if λ crosses the "Stokes" lines separating them, the fundamental system approximated by the author's expressions changes abruptly into another, not completely specified, fundamental system. The method used is similar to that of Langer in that its basic idea is a comparison with a "related" differential equation whose coefficients differ (in a whole neighborhood of s_0) only unessentially from those of the given differential equation. *W. Wasow.*

Rapoport, I. M. On the asymptotic behavior of solutions of linear differential equations. Doklady Akad. Nauk SSSR (N.S.) 78, 1097–1100 (1951). (Russian)

Let $w_i(t)$, $i = 1, \cdots, n$, be integrable over any finite interval (t_0, t_1) and let Re $[w_i(t) - w_j(t)]$ be of fixed sign for large t and all $i, j, i \neq j$. Let $c_{ij}(t)$, $i, j, = 1, \cdots, n$, be integrable (t_0, ∞). Then the system of differential equations

$$\frac{dy_i}{dt} = w_i(t)y_i + \sum c_{ij}y_j$$

has for each $k = 1, 2, \cdots, n$ a solution

$$y_{ik}(t) = \exp\left(\int_{t_0}^{t} w_k(\tau)\eta_{ik}(\tau)d\tau\right)$$

where $\eta_{ik}(t)$ is continuous and $\eta_{ik}(\infty) = 0$, $i \neq k$, $\eta_{kk}(\infty) = 1$.

The vector system $dx/dt = (A_0(t) + A_1(t))x$ is considered with $A_0'(t)$ and $A_1(t)$ integrable over $(0, \infty)$. [Here apparently the author is unfamiliar with the treatment of this case by the reviewer, Duke Math. J. 15, 111–126 (1948); these Rev. 9, 509.] Other results are stated. *N. Levinson* (Cambridge, Mass.).

Letov, A. M. Bounds for the smallest characteristic value of a class of regulating systems. Akad. Nauk SSSR. Prikl. Mat. Meh. 15, 591–600 (1951). (Russian)

The author considers the non-linear system

$$du_k/dt = \sum_{j=1}^{n} a_{kj}u_j + u_k v, \quad k = 1, 2, \cdots, n, \quad dv/dt = f\left(\sum_{k=1}^{n} c_k u_k - v\right),$$

and gives bounds on the smallest characteristic value of the associated linear system under certain assumptions concerning f. *R. Bellman* (Stanford University, Calif.).

Makarov, S. M. An investigation of the characteristic equation of a linear system of two equations of the first order with periodic coefficients. Akad. Nauk SSSR. Prikl. Mat. Meh. 15, 373–378 (1951). (Russian)

The classical result of Lyapounov for $\ddot{x} + p(t)x = 0$ is generalized. Thus, if $\dot{x}_1 = p_{12}(t)x_2$, $\dot{x}_2 = -p_{21}(t)x_1$, where p_{12} and p_{21} are continuous real periodic functions of real period ω in t and are non-negative (and not identically zero), then $\int_0^\omega p_{12}dt \cdot \int_0^\omega p_{21}dt \leq 4$ implies the characteristic roots are imaginary and of modulus one. [Remark. If s is defined by $ds/dt = p_{12}(t)$ then the above system yields the classical case $d^2x_1/ds^2 + \phi(s)x_1 = 0$. By using the result of Borg [Amer. J. Math. 71, 67–70 (1949); these Rev. 10, 456] and observing that the continuity requirement in $\phi(s)$ can be replaced by integrability, it follows that a more general result is valid than the above, namely,

$$\int_0^\omega p_{12}dt \cdot \int_0^\omega |p_{21}|dt \leq 4,$$

where now $p_{21}(t) \not\equiv 0$ and $\int_0^\omega p_{21}dt > 0$.] The case $\dot{x}_i = \sum_1^2 p_{ij}x_j$, $i = 1, 2$, is discussed and with the sum of the characteristic roots $\rho_1 + \rho_2 = 2A$ it is shown that

$$2A > \exp\left(-\int_0^\omega p_{11}dt\right) + \exp\left(-\int_0^\omega p_{22}dt\right)$$

where p_{ij} are assumed to be non-negative and not identically zero. *N. Levinson* (Cambridge, Mass.).

Sobol', I. M. Investigation with the aid of polar coordinates of the asymptotic behavior of solutions of a linear differential equation of the second order. Mat. Sbornik N.S. 28(70), 707–714 (1951). (Russian)

The equation $\ddot{x} + 2p(t)\dot{x} + q(t)x = 0$ is considered. In case constants p and q exist such that $q - p^2 = c^2 > 0$ and $\int_0^t |p(t) - p|dt = o(t)$ as $t \to \infty$ and similarly for $q(t)$ then it is shown easily that the successive zeros t_n of a real solution satisfy $t_n \sim \pi n/c$. If $p(t)$ and $q(t)$ are of bounded variation over (a, ∞) then $x(t) = O(\exp[-\int_a^t p(t)dt])$. [Here much more is known. From a result of the reviewer [Duke Math. J. 15, 111–126 (1948); these Rev. 9, 509] it follows that there exist independent solutions x_1 and x_2 with $x_1(t) \sim \exp(\int_a^t \{[p^2(t) - q(t)]^{\frac{1}{2}} - p(t)\}dt)$ and x_2 similarly with

a minus before the square root.] The case $\ddot{x}+\omega^2(t)x=0$ is considered with various hypotheses in $\dot{\omega}/(2\omega^2)$. [The results when $\dot{\omega}/\omega^2$ is of bounded variation on (a, ∞) can be improved by using the result of the reviewer mentioned above.] Further results are given. *N. Levinson.*

Hartman, Philip. The number of L^2-solutions of $x''+q(t)x=0$. Amer. J. Math. **73**, 635–645 (1951).

A number of non-equivalent sufficient conditions for (*) $x''+q(t)x=0$ to be in the limit point case have recently been given. The author obtains a sufficient condition which he is then able to show includes the earlier conditions. Let $N(t)$ be the number of zeros in $(0, t)$ of a non-trivial solution of (*). The author's criterion is that there exist a positive continuous function $Q(t)$ defined for large t, of bounded variation on any finite interval and satisfying

$$\limsup_{t\to\infty}\left\{2\int^t Q^{-1}(s)ds - \int^t Q^{-2}(s)dN(s) - Q^{-2}(t) - \int^t |dQ^2(s)|\right\} = \infty.$$

N. Levinson (Cambridge, Mass.).

Bremmer, H. The W.K.B. approximation as the first term of a geometric-optical series. Comm. Pure Appl. Math. **4**, 105–115 (1951).

The author discusses solutions of the equation

$$y''+k^2(x)y=0,$$

where the wave number $k(x)$ is a variable function of x for $x>0$ and a constant k_0 for $x<0$. He assumes that a plane wave, $\exp(ik_0x)$, arrives from the space $x<0$ and travels upwards in the direction of increasing x, while the non-homogeneous medium is replaced by a set of finite layers, $x_s<x<x_{s+1}$, in each of which $k(x)$ has a constant value k_{s+1}. At the boundary of each layer the incident wave is refracted upwards and reflected downwards. The familiar W.K.B. approximation is obtained by taking the sum of all the refracted waves at each successive boundary, with no internal reflections, and passing to the limit of infinitely thin layers. This suggests that a second approximation can be obtained by considering the sum of the waves obtained after one reflection, and so on. Thus the solution of the equation is represented as the sum of two sets of waves: u_{2N} traveling upwards and u_{2N+1} downwards, the subscripts denoting the number of reflections. Simple recurrence relations are obtained for the u_N's, and the total upgoing and downgoing waves are shown to satisfy certain integral equations. The theory is applied to obtain the successive reflected waves in a medium with $k(x)=k_0/(1+2R_0k_0x)$, $x>0$ and R_0 constant.

M. C. Gray (Murray Hill, N. J.).

Vogel, Théodore. Les méthodes topologiques de discussion des problèmes aux oscillations non linéaires. Ann. Télécommun. **6**, 2–10 (1951).

Expository paper.

Ascoli, Guido. Ricerche asintotiche sopra una classe di equazioni differenziali non lineari. Ann. Scuola Norm. Super. Pisa (3) **5**, 1–28 (1951).

In the differential equation $\ddot{x}+x=f(x, 1/t)$ let $f(x, u)$ be analytic near $x=u=0$, and let $f(0, u)=f(x, 0)=0$. The main result of the paper is that this equation possesses solutions that remain bounded as $t\to\infty$, and that every such bounded solution $x(t)$ admits an asymptotic representation of the form

$$x(t)=\gamma\sin\left(t-\frac{\phi(t)}{2\pi}\log t-\mu\right)+O\left(\frac{1}{t}\right),$$

where γ and μ are constants and

$$\frac{\phi(t)}{2\pi}=\frac{1}{2\pi\gamma}\int_0^{2\pi} f_u(\gamma\sin\sigma, 0)\sin\sigma d\sigma.$$

The existence of bounded solutions is established by transforming the differential equation into a canonical system and studying the stability of the stationary solution of that system. In the proof of his asymptotic formula the author makes essential use of Sturm's theorems on the zeros of linear second order equations. Various extensions and applications of this result are discussed. Among these are the calculation of the second order term in the asymptotic representation, an asymptotic formula for the case that the right member is of the form $f(x, 1/\sqrt{t})$, and asymptotic expressions for the zeros of the bounded solutions.

W. R. Wasow (Los Angeles, Calif.).

Magnus, K. Erzwungene Schwingungen des linearen Schwingers bei nichtharmonischer Erregung. Z. Angew. Math. Mech. **31**, 324–329 (1951).

The periodic solution of the differential equation

$$m\ddot{x}+d\dot{x}+cx=f(t)$$

is studied for three special forms of $f(t)$, corresponding to the following forms of excitation: a) a force of constant amount reversing periodically its direction; b) periodic instantaneous impulses in opposite directions; c) periodic instantaneous impulses in the same direction. Instead of the methods of Fourier analysis the author uses the fact that, except at the points of discontinuity of $f(t)$, the solution of the problem solves also the homogeneous equation. The condition that the solution must be continuously differentiable permits then an elementary determination of the constants of integration. The resonance curves and several limit cases are discussed in some detail. *W. R. Wasow.*

Minorsky, Nicolas. Sur le pendule entretenu par un courant alternatif. C. R. Acad. Sci. Paris **233**, 728–729 (1951).

If a pendulum containing a piece of iron is placed near an alternating current in a coil, it may begin to oscillate. The differential equations of this system are

$$d[L(\theta)i]/dt+si=E\sin\omega t,\quad J\ddot{\theta}+D\dot{\theta}+C\theta=d[\tfrac{1}{2}L[\theta]i^2]/d\theta,$$

where $L[\theta]$ is mildly non-linear, and even or odd, depending on the physical arrangement. In order to find approximately a periodic solution of this system the author approximates θ tentatively by $\theta=\theta_0\cos\Omega t$, inserts this into the first equation and calculates approximately the periodic solution of the resulting differential equation for i. After resubstitution of this result into the second equation, the latter is seen to be, for a particular value of Ω, of a type which was shown by the author to possess a stable periodic solution [same C. R. **232**, 1060–1062 (1951); these Rev. **12**, 611]. The frequency ω of the current will, in general, not be in a rational relation with Ω. *W. R. Wasow.*

Graffi, D. Forced oscillations for several nonlinear circuits. Ann. of Math. (2) **54**, 262–271 (1951).

Consider the system

$$L_1\ddot{x}_1+M\ddot{x}_2+g_1'(x_1)\dot{x}_1+x_1/C_1=e_1(t),$$
$$M\ddot{x}_1+L_2\ddot{x}_2+g_2'(x_2)\dot{x}_2+x_2/C_2=e_2(t).$$

Assume g_i' continuous, $g_i(0)=0$, $\lim_{|x|\to\infty} g_i(x)/x=R_i>0$, e_i periodic of period T, $C_i>0$, $L_i>0$, $M^2<L_1L_2$,

$$M^2(R_1C_1+R_2C_2)^2<4R_1C_1R_2C_2L_1L_2.$$

Then a periodic solution of period T exists. The proof is based on Brouwer's fixed point theorem. *J. L. Massera.*

Pucci, Carlo. Sulla maggiorazione dell'integrale di una equazione differenziale lineare ordinaria del secondo ordine. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **10**, 300–306 (1951).

Several bounds for the solution of $y''+A(x)y=f(x)$ satisfying $y'(a)=y'(b)=0$ are found; these bounds depend on the bounds and integrals of A, f and the eigenvalues of $y''+\lambda Ay=0$. *J. L. Massera* (Montevideo).

Hartman, Philip. On the eigenvalues of differential equations. Amer. J. Math. **73**, 657–662 (1951).

The differential equation (1) $x''+(\lambda-q)x=0$ is considered over $0\leqq t<\infty$ with $q=q(t)$ real and continuous and λ real. Let $q(t)\to\infty$ as $t\to\infty$. Subject to a real homogeneous boundary condition at $t=0$, a spectrum $\lambda_0<\lambda_1<\cdots$ is determined. If the conditions $q'>0$, $q''\geqq 0$ and $q''=o(q'^{4/3})$ as $t\to\infty$ are met, Milne has shown that as $n\to\infty$, (2) $\pi n\sim\int_0^r(\lambda_n-q)^{\frac{1}{2}}dt$ where $r=r_n$ is the value of t for which $q(t)=\lambda_n$. The author proves that (2) holds if $q(t)$ is an increasing function satisfying, as $t\to\infty$,

$$\inf\left[(q(v)-q(u))\Big/\int_u^v s^{-3}ds\right]\to\infty$$

where $t\leqq u<v<\infty$. But (2) need not hold if $q'(t)$ exists and $\liminf q'(t)t^3<\infty$. *N. Levinson* (Cambridge, Mass.).

Hartman, Philip. On bounded Green's kernels for second order linear ordinary differential equations. Amer. J. Math. **73**, 646–656 (1951).

Let $q(t)$ be real and continuous for $0\leqq t<\infty$ and let $q(t)<C^2$. Consider (*) $x''+(\lambda+q(t))x=0$ over $(0,\infty)$. Let λ be a real number not in the essential spectrum of (*). Then the author proves that there exists an $\epsilon=\epsilon(\lambda)>0$ and a solution of (*), $x=z(t)$, satisfying $\limsup(z^2+z'^2)e^{\epsilon t}<\infty$, as $t\to\infty$. Also every solution $x=y(t)$ of (*) independent of $z(t)$ satisfies $\liminf(y^2+y'^2)e^{-\epsilon t}>0$. *N. Levinson.*

Levinson, Norman. Addendum to "A simplified proof of the expansion theorem for singular second order linear differential equations." Duke Math. J. **18**, 719–722 (1951).

Acknowledging that a proof in the paper to which this is an addendum [same J. **18**, 57–71 (1951); these Rev. **12**, 828] has been justifiably questioned, the author supplies here another proof. The theorem, in which $\varphi(x,u)$ is a principal solution of a Sturm-Liouville differential equation, and $\rho(u)$ a certain monotone function, is the following: "If $G(u)$ is measurable B, and

$$\int_{-\infty}^{\infty}|G(u)|^2d\rho(u)<\infty,$$

then

$$\underset{a\to\infty}{\text{l.i.m.}}\int_{-a}^{a}G(u)\varphi(x,u)d\rho(u)=f(x)$$

exists; and with

$$g(u)=\underset{a\to\infty}{\text{l.i.m.}}\int_0^a f(x)\varphi(x,u)dx,$$

the relation

$$\int_{-\infty}^{\infty}|G(u)-g(u)|^2d\rho(u)=0,$$

is fulfilled." *R. E. Langer* (Madison, Wis.).

Putnam, C. R. On the least eigenvalue of Hill's equation. Quart. Appl. Math. **9**, 310–314 (1951).

The equation (*) $x''+(\lambda+f(t))x=0$ is considered with f real, continuous and periodic of period 1 for $-\infty<t<\infty$. Let μ be the real number such that (*) is oscillatory for $\lambda>\mu$ but not for $\lambda<\mu$. Let $f(t)$ have Fourier series $\sum_{-\infty}^{\infty}c_ne^{2\pi int}$. Then for any $N\geqq 1$ the author proves $\mu\leqq\pi^2N^2-c_0+\mathrm{Re}\,(c_N)$. If $-f(t)=f(t+c)$ for some real c then $\mu\leqq\pi^2N^2-c_0-|\mathrm{Re}\,(c_N)|$. *N. Levinson.*

Levitan, B. M. The application of generalized displacement operators to linear differential equations of the second order. Amer. Math. Soc. Translation no. **59**, 135 pp. (1951).

Translated from Uspehi Matem. Nauk (N.S.) **4**, no. 1(29), 3–112 (1949); these Rev. **11**, 116.

Schäfke, Friedrich Wilhelm. Zur Parameterabhängigkeit bei gewöhnlichen linearen Differentialgleichungen mit singulären Stellen der Bestimmtheit. Math. Nachr. **6**, 45–50 (1951).

The paper is concerned with the linear equation

$$F(y)=\lambda G_1(y)+\lambda^2G_2(y)+\cdots+\lambda^kG_k(y),$$

in which F is a differential form of order n, $G_1,\cdots,G_k$ are such forms of orders less than n, and λ is a parameter. If $x=a$ is a regular singular point at which the indicial equation is free from λ, and r is an exponent which is not less than any other exponent by an integer, the solution with the exponent r is an entire function of λ. An upper bound for the order of this function is established, and facts regarding the characteristic values of some boundary problems based on the differential equation are deduced. The paper has close connection with that reported in an earlier paper [Math. Nachr. **3**, 20–39 (1949); these Rev. **11**, 594]. *R. E. Langer* (Madison, Wis.).

✶Saltikow, N. N. Problèmes actuels de la théorie moderne d'équations aux dérivées partielles du premier ordre à une fonction inconnue. Premier Congrès des Mathématiciens et Physiciens de la R.P.F.Y., 1949. Vol. II, Communications et Exposés Scientifiques, pp. 23–31. Naučna Knjiga, Belgrade, 1951. (Serbo-Croatian. French summary)

Myškis, A. D., and Abolinya, V. Ė. On uniqueness of solution of a mixed problem for systems of partial differential equations. Doklady Akad. Nauk SSSR (N.S.) **80**, 533–536 (1951). (Russian)

The authors prove uniqueness theorems for the generalised telegraphic system

$$(1)\quad \begin{aligned}&\mathbf{A}_1\partial\mathbf{i}/\partial t+\sum_{k=1}^n\mathbf{B}_{1k}\partial\mathbf{i}/\partial x_k+\sum_{k=1}^n\mathbf{C}_{1k}\partial\mathbf{u}/\partial x_k+\mathbf{D}_1\mathbf{i}+\mathbf{F}_1\mathbf{u}=\mathbf{h}_1,\\&\mathbf{A}_2\partial\mathbf{u}/\partial t+\sum_{k=1}^n\mathbf{B}_{2k}\partial\mathbf{u}/\partial x_k+\sum_{k=1}^n\mathbf{C}_{2k}\partial\mathbf{i}/\partial x_k+\mathbf{D}_2\mathbf{u}+\mathbf{F}_2\mathbf{i}=\mathbf{h}_2,\end{aligned}$$

where the unknowns $\mathbf{i}$, $\mathbf{u}$ are mth order column matrices, as also $\mathbf{h}_1$, $\mathbf{h}_2$, and $\mathbf{A}_1, \cdots, \mathbf{F}_2$ are mth order square matrices, all being functions of x and t, where $x\equiv(x_1, \cdots, x_n)\epsilon G$, a certain bounded n-dimensional region with boundary Γ, and $0\leqq t<T$ $(0<T\leqq\infty)$. Furthermore, the matrices $\mathbf{A}$ and $\mathbf{B}$ are to be symmetric and there are continuity restrictions. Putting $\mathbf{i}=e^{\alpha t}\mathbf{j}$ and $\mathbf{u}=e^{\alpha t}\mathbf{v}$ for any constant α and assuming further that $\mathbf{C}_{12}$ is the transpose of $\mathbf{C}_{21}$ they find for the homogeneous case $(\mathbf{h}_1=\mathbf{h}_2=0)$ an integral identity (too long to reproduce) bearing some resemblance to an energy equation. They deduce a theorem, that if further $\mathbf{B}_{12}|_\Gamma=0$, and $\mathbf{A}_1\geqq0$, $\mathbf{A}_2\geqq0$, then (1) has at most one solution satisfying the boundary conditions $\mathbf{i}|_{t=0}=f(x)$, $\mathbf{u}|_{t=0}=\phi(x)$ $(x\epsilon G)$, $\mathbf{u}|_{x\epsilon\Gamma}=\psi(x, t)$ $(0\leqq t<T)$, provided that for some α a certain quadratic form involving α is negative definite. This theorem is amplified by seven remarks. The first relates to the analogous case in which $\mathbf{B}_{21}|_\Gamma=0$ instead of $\mathbf{B}_{12}|_\Gamma=0$. The second says that if $\mathbf{A}_1>0$, $\mathbf{A}_2>0$, then the condition of negative definiteness is satisfied for large α. The third gives more complicated conditions with the same effect; this includes a previous uniqueness theorem of N. A. Brazma and A. D. Myškis [Akad. Nauk SSSR. Prikl. Mat. Meh. **15**, 495–500 (1951); these Rev. **13**, 351]. The fourth generalises the boundary condition $u|_\Gamma=\psi$. The fifth suggests relaxation of the continuity restrictions; cf. here A. D. Myškis [Mat. Sbornik N.S. **26(68)**, 341–344 (1950); these Rev. **12**, 609]. The sixth compares cases, e.g. analytic equations, in which $\mathbf{A}_1\geqq0$, $\mathbf{A}_2\geqq0$ can be relaxed to $|\mathbf{A}_1|\neq0$, $|\mathbf{A}_2|\neq0$. Finally they illustrate their theorem by constructing an example in which the uniqueness fails. *F. V. Atkinson* (Ibadan).

Moisil, Gr. C. On the theory of characteristics of systems of partial differential equations. An. Acad. Repub. Pop. Române. Secţ. Şti. Mat. Fiz. Chim. Ser. A. **2**, 639–667 (1949). (Romanian. Russian and French summaries)

In this paper the author studies the characteristics of systems of m partial differential equations in m unknown functions φ_i of the $n+1$ independent variables $x_1, \cdots, x_n, y$, of the form

$$\frac{\partial\varphi_i}{\partial x_\mu}=\sum_j\alpha_{\mu ij}\frac{\partial\varphi_j}{\partial y}, \quad \mu=1, \cdots, m,$$

where $\alpha_{\mu ij}=\alpha_{\mu ij}(x_1, \cdots, x_m, y)$, the compatibility equations for the system being assumed to be satisfied. As a special case this system contains the system of partial differential equations which takes the place of the customary Cauchy-Riemann equations in the theory of analytic functions $f=a(x, y)+jb(x, y)+j^2c(x, y)+j^3d(x, y)$ of the hypercomplex variable $x+jy$, where $(1+j^2)^2=0$ [L. Sobrero, Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. **6**, 1–64 (1934)]. Another system of partial differential equations, which contains as a special case the Cauchy-Poincaré equations for functions of two complex variables, is also treated.

J. B. Dias (College Park, Md.).

Loewner, Charles. A transformation theory of the partial differential equations of gas dynamics. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2065**, 56 pp. (1950).

The author investigates a class of transformations of Baecklund type (see (B)) which transform the system (A) (see below) of linear partial differential equations of first order in two independent variables, s, t. (It is assumed that the equations under consideration cannot be transformed into each other by simple point transformations.) Corresponding solutions of the two systems are derived from each other by a process of solving systems of ordinary differential equations. A system (A) can be written in the form

$$\text{(A)} \qquad \zeta_s=H\zeta_t.$$

Here ζ is a one-column matrix, and H is a given 2×2 matrix, whose elements are functions of s and t. The Baecklund transformations are written in the form

$$\text{(B)} \qquad \zeta_s^*=W\zeta_s+A\zeta+C\zeta^*, \quad \zeta_t^*=W\zeta_t+B\zeta+D\zeta^*,$$

the coefficients W, A, B, C, D being matrix functions of s and t, and ζ^* and ζ are column vectors. The author determines the necessary and sufficient conditions on the coefficients A, B, C, D in order that there exist two systems of partial differential equations $\zeta_s=H\zeta_t$ and $\zeta_s^*=H^*\zeta_t^*$, so that each solution of the first system is transformed by (B) into solutions of the second system. In this way he can decide whether or not a given system can be transformed in a desired simpler system. In particular, he finds that a certain twelve-parameter family of equations can be transformed into the Cauchy-Riemann equations. Further, he finds a several-parameter family of equations which can be transformed into a system which is equivalent to the Tricomi equation. The author emphasizes that the considered transformations do not form a group, and therefore by composition new transformations can be obtained, which transformations allow finding higher parameter families of systems of equations which can be transformed to some given canonical form. Applications to the theory of compressible fluid flows are given. *S. Bergman.*

Biegelmeier, Gottfried. Ein Beitrag zur klassischen Diffusionstheorie. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa **159**, 161–172 (1950).

A cube A containing a diffusing substance of concentration α is placed in a medium of concentration β. (Either α or β is zero, and $\alpha\neq\beta$.) Experimentally it is noted that the diffusing substance soon tends to occupy a spherical-shaped region. The author then checks the experimental and theoretical results. Mathematically speaking, the following boundary value problem is solved by aid of the fundamental solution of the heat equation. Find $u=u(x, y, z; t)$ such that $u_t=u_{xx}+u_{yy}+u_{zz}$, $t>0$, while at time $t=0$ the function $u=\alpha$ or β according as the point (x, y, z) is interior or exterior to the cube A. (Similar problems are handled for one and two space variables.) *F. G. Dressel.*

Thiruvenkatachar, V. R. On a differential equation allied to the equation of diffusion. Half-Yearly J. Mysore Univ. Sect. B., N.S. **5**, 31–34 (1944).

Let $a(t)x^2+2h(t)xy+b(t)y^2$ be a positive definite form for all $t\geqq0$, and consider the parabolic equation

$$\text{(1)} \qquad u_t=au_{xx}+2hu_{xy}+bu_{yy}.$$

Starting with elementary solutions, and using the method of superposition, the paper generates a fundamental solution U_1 of (1). The function U_1 represents the solution for an instantaneous source of unit strength at the origin at time $t=0$. The author generates another function U_2 which he states is a solution of (1) and the singularity at time $t=0$ of the solution is the elliptic cylinder $b(0)x^2-2h(0)xy+a(0)y^2=k$, k a constant. The reviewer was unable to verify that U_2 satisfies equation (1). *F. G. Dressel* (Durham, N. C.).

Linelkin, P. S. On the cooling of the surface layer of the sea. Doklady Akad. Nauk SSSR (N.S.) **80**, 205–208 (1951). (Russian)

A method of successive approximation solution for the one-dimensional non-steady heat conduction equation is

outlined for the case where the effective thermal diffusivity is a product of time dependent and distance dependent functions. The procedures and conclusions offer nothing not readily available from existing sources. *N. A. Hall.*

Datta Majumdar, Sudhansu. A note on the differential equation of the Clusius column for separation of isotopes. Science and Culture 15, 329 (1950).

The differential equation in question is the following: $Mc_t = -H(1-2c)c_x + Kc_{xx}$, where M, H, and K are constants.

Linés Escardó, Enrique. Solution in finite form of Cauchy's problem on an arbitrary hypersurface for the wave equation with an arbitrary number of variables and for other notable equations of hyperbolic type with constant coefficients. Collectanea Math. 2, 3–86 (1949). (Spanish)

The present paper is divided into three long chapters. Chapter I contains the solution in finite form of Cauchy's problem for an arbitrary hypersurface for the wave equation in any number of variables, i.e. the problem of finding the function $u(t, x_1, \cdots, x_n)$ satisfying the wave equation

$$(*) \qquad \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} - \cdots - \frac{\partial^2 u}{\partial x_n^2} = f(t, x_1, \cdots, x_n),$$

when u is required to satisfy the Cauchy conditions

$$[u(x_i, t)]_\Gamma = [u(x_i, t)]_{t=\psi} = \varphi_0(x_i),$$

$$\left[\frac{\partial u}{\partial t}(x_i, t)\right]_\Gamma = \left[\frac{\partial u}{\partial t}(x_i, t)\right]_{t=\psi} = \varphi_1(x_i),$$

where Γ is a regular portion of a non-characteristic surface of (*), given by the equation $t = \psi(x_1, \cdots, x_n)$, and φ_0 and φ_1 are given functions on Γ. At first, all functions involved are supposed to be analytic. The author employs the method of linear operators of L. Fantappiè [Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 2, 948–956 (1941); these Rev. 8, 210; Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. 1, 25–57 (1930)] and obtains first an integrodifferential equation which is equivalent to the Cauchy problem started out with. The solution of the integrodifferential equation reduces to the calculation of the operator whose "projective indicatrix" [L. Fantappiè, Ann. Mat. Pura Appl. (4) 22, 181–289 (1943); these Rev. 8, 589] is:

$$p(\alpha_1, \alpha_2, \cdots, \alpha_n) = \left[1 - \sum_{i=1}^n \alpha_i^2\right]^{-1}.$$

The solution of the Cauchy problem is given in various forms, several special cases are studied, and Huyghens' principle is discussed. Cauchy's problem in the real domain is then considered. Chapter II deals with the solution of Cauchy's problem for certain hyperbolic partial differential equations and systems with constant coefficients, e.g. the equation

$$\prod_{j=1}^k \left(\frac{\partial^2}{\partial t^2} - \sum_{i=1}^n a^2{}_{ij} \frac{\partial^2}{\partial x_i^2}\right) u = f(t, x_1, \cdots, x_n),$$

and the system of partial differential equations for light propagation in an uniaxial crystalline medium [see F. Bureau, C. R. Acad. Sci. Paris 226, 1331–1333 (1948); these Rev. 9, 591]. Chapter III deals with the calculation of certain projective products, starting with particular solutions of partial differential equations with constant coefficients, and also with the solution of Cauchy's problem, Cauchy data given on the plane $t=0$, for the equation

$$a_{00}\frac{\partial^4 u}{\partial t^4} + \sum_{i,j=1}^n a_{ij}\frac{\partial^4 u}{\partial x_i^2 \partial x_j^2} + \sum_{i=1}^n a_{0i}\frac{\partial^4 u}{\partial x_i^2 \partial t^2} = f(t, x_1, \cdots, x_n).$$

J. B. Diaz (College Park, Md.).

Èĭdus, D. M. On a mixed problem of the theory of elasticity. Doklady Akad. Nauk SSSR (N.S.) 76, 181–184 (1951). (Russian)

Consider a finite domain Ω in three-dimensional (x_1, x_2, x_3)-space, with a piecewise smooth boundary Γ which consists of three parts $\Gamma_1, \Gamma_2, \Gamma_3$. A certain mixed problem of the theory of elasticity consists in the determination of the displacement vector u, which satisfies the equation

$$\mu \Delta u + (\mu + \lambda) \text{ grad div } u + f = 0$$

in Ω, and also the following boundary conditions

$$u|_{\Gamma_1} = 0, \quad u_n|_{\Gamma_2} = 0, \quad t_\tau|_{\Gamma_2} = 0, \quad t|_{\Gamma_3} = 0,$$

where n and τ denote normal and tangential components, respectively, and t is the stress vector. Let D denote the class of all functions $u = (u_1, u_2, u_3)$ with continuous first partial derivatives in Ω, and such that

$$H(u) = \int_\Omega \sum_{i=1}^3 u_i^2 d\Omega < \infty, \quad D(u) = \int_\Omega \sum_{i,k=1}^3 \left(\frac{\partial u_i}{\partial x_k}\right)^2 d\Omega < \infty.$$

Further, let $\dot{D}_1$ denote the subset of D consisting of all functions u satisfying $u|_{\Gamma_1} = 0$. Introduce a norm in D, by means of the definition

$$\|u\| = [H(u) + D(u)]^{\frac{1}{2}},$$

and let $\mathring{D}_1$ be the closure (completion) of the space $\dot{D}_1$ with respect to this norm. According to S. G. Mihlin [Direct Methods in Mathematical Physics, Moscow-Leningrad, 1950] in order to solve the above mixed problem one need only establish that there exist positive numbers α and β such that, for any u in $\mathring{D}_1$, the following two inequalities hold:

$$(1) \qquad H(u) \leqq \alpha D(u),$$

$$(2) \qquad D(u) \leqq \beta S(u),$$

where

$$S(u) = \int_\Omega \sum_{i,k=1}^3 s^2{}_{ik} d\Omega, \quad s_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right).$$

Inequality (1) was proved by S. L. Sobolev [Mat. Sbornik N.S. 2(44), 467–500 (1937)]. Using results of K. O. Friedrichs [Ann. of Math. (2) 48, 441–471 (1947); these Rev. 9, 255], the present author proves the inequality (2).

J. B. Diaz (College Park, Md.).

Ripianu, Dumitru. Existence theorems for linear hyperbolic partial differential equations of order n. Acad. Repub. Pop. Române. Bul. Şti. Ser. Mat. Fiz. Chim. 2, 109–118 (1950). (Romanian. Russian and French summaries)

Discussion of certain initial value problems associated with the partial differential equation $\partial^n u/\partial x_1 \partial x_2 \cdots \partial x_n = F$ where u is a function of $x_1, \cdots, x_n$ and F is a linear function (with variable coefficients) of u and of the partial derivatives of u up to and including order $n-1$. *A. Erdélyi.*

Difference Equations, Special Functional Equations

Conti, Roberto. Un teorema di confronto per le equazioni alle differenze finite, lineari, del 2° ordine. Boll. Un. Mat. Ital. (3) 6, 208–213 (1951).

The author proves the following theorem related to difference equations which is analogous to classical theorems of Sturm related to differential equations. Let $y(x)$, $z(x)$ be two continuous solutions of the respective equations

$$\Delta[\theta(x)\Delta y(x)]-Q(x)y(x+h)=0,$$
$$\Delta[\theta_1(x)\Delta z(x)]-Q_1(x)z(x+h)=0,$$

for $\alpha\leqq x\leqq\beta=\alpha+nh$, $n\geqq 3$. Further let y and z be linearly independent on the set of points $x=\alpha+ih$, $i=0, 1, \cdots, n-1$, and satisfy the conditions 1) $y(\alpha)=y(\beta)=0$,

2) $$\lim_{x\to\alpha+}(y^2/z)=\lim_{x\to\beta-}(y^2/z)=0,$$

and 3) $y(\alpha+ih)\neq 0$, $i=1, 2, \cdots, n-1$. Then if $\theta(x)$, $\theta_1(x)$, $Q(x)$, $Q_1(x)$ are continuous on $\alpha\leqq x\leqq\beta$ and satisfy the conditions

$$Q(\alpha+ih)\geqq Q_1(\alpha+ih),\quad \theta(\alpha+ih)\geqq\theta_1(\alpha+ih)>0$$

for $i=0, 1, \cdots, n-1$, there exists at least one point of the interval $\alpha<x<\beta$ at which $z(x)$ vanishes.

P. E. Guenther (Cleveland, Ohio).

Gnanadoss, Adaikalam A. Linear difference equations with periodic coefficients. Proc. Amer. Math. Soc. 2, 699–703 (1951).

The author proves the following theorem: A linear difference equation of order n with periodic coefficients of common period λ can be transformed into a linear difference equation of order $n\lambda$ with constant coefficients. This is a generalization of a theorem proved by the reviewer for equations of the second order. *T. Fort* (Athens, Ga.).

Esipovič, E. M. On stability of solutions of a class of differential equations with retarded argument. Akad. Nauk SSSR. Prikl. Mat. Meh. 15, 601–608 (1951). (Russian)

Differential-difference systems of the form

$$dy(t)/dt=Ay(t)+By(t-h)$$

lead to transcendental equations of the form

$$P(z)+Q(z)e^{-z}=0,$$

where P and Q are polynomials, for the determination of the characteristic roots. The author presents some general results on the location of the roots of this equation and treats some particular equations by way of illustration.

R. Bellman (Stanford University, Calif.).

Alaci, V. Une classe d'équations fonctionnelles. An. Acad. Repub. Pop. Romåne. Secţ. Şti. Mat. Fiz. Chim. Ser. A. 3, 461–477 (1950). (Romanian. Russian and French summaries)

A function $f(x, y, z)$ is called generalized pseudo-homogeneous, in case there exist functions Φ, φ_1, φ_2, φ_3 of t, such that $f\{\varphi_1(t)x, \varphi_2(t)y, \varphi_3(t)z\}=\Phi(t)f(x, y, z)$ holds identically in t, x, y, z. This is a generalization of pseudo-homogeneous functions [the author, Bull. Sci. École Polytech. Timişoara 11, 6–13 (1943); these Rev. 9, 16; Revista Mat. Timişoara 3, no. 1, 3–4 (1924)]. It has been shown [2d cited reference] that $f(x, y, z)$ satisfies also the functional equation (*) $\varphi_1^{x\partial f/\partial x}\cdot\varphi_2^{y\partial f/\partial y}\cdot\varphi_3^{z\partial f/\partial z}=\Phi^f$. Conditions on Φ, φ_i $(i=1, 2, 3)$ are established, so that (*) should have solutions independent of t and the corresponding solutions are found; they are generalized pseudo-homogeneous functions. The definition and properties generalize immediately to functions of any number of independent variables. Reference is made to an investigation of the properties of generalized pseudo-homogeneous functions in a paper to be published.

E. Grosswald (Princeton, N. J.).

Kuwagaki, Akira. Sur l'équation fonctionnelle: $$f(x+y)=R\{f(x), f(y)\}.$$ Mem. Coll. Sci. Univ. Kyoto Ser. A. Math. 26, 139–144 (1951).

In the above equation $R(u, v)$ is a given rational function and $f(x)$, to be determined, is only required to be continuous. It is shown that if the equation has a single-valued non-constant continuous solution on an interval, then $R(u, v)=\{auv+buv+c\}/\{puv+quv+r\}$, where $a, \cdots, r$ are constants with $a:b:c\neq p:q:r$ and $\mathrm{Rank}\left(\begin{smallmatrix} b & a-q & p\\ c & b-r & q\end{smallmatrix}\right)=1$; and that in consequence the given equation can be reduced by a suitably chosen linear fractional transformation to one of the two forms $f^*(x+y)=f^*(x)f^*(y)$, $f^*(x+y)=f^*(x)+f^*(y)$. Since the continuous solutions of these equations are known, one thus obtains the continuous solutions of the original equation. If R_1, R_2 are rational functions of the form of R above (and subject to the same conditions on the coefficients), the equation $f\{R_1(x, y)\}=R_2\{f(x), f(y)\}$ is shown to be transformable into the equation of the title, and is thus solvable. *I. M. Sheffer* (State College, Pa.).

Aczél, János. Über Funktionalgleichungen mehrerer Veränderlichen. I. Elementare Lösungsmethoden für Funktionalgleichungen mehrerer Veränderlichen. Mat. Lapok 2, 99–117 (1951). (Hungarian. Russian and German summaries)

This is an exposition of the author's results according to which the continuous, increasing solutions of

$$m(m(x, y), m(\xi, \eta))=m(m(x, \xi), m(y, \eta))$$

are $m(x, y)=f^{-1}(p_1f(x)+p_2f(y)+p)$, where $f(x)$ is continuous and increasing [Bull. Amer. Math. Soc. 54, 392–400 (1948); these Rev. 9, 501]; the continuous and increasing solutions of

$$F(F(x, y), z)=F(x, F(y, z))$$

are $F(x, y)=f^{-1}(f(x)+f(y))$, where $f(x)$ is continuous and increasing [Bull. Soc. Math. France 76, 59–64 (1949); these Rev. 10, 685]. The author gives the main ideas of the proofs, giving them in the order in which he got them.

P. Erdös (Aberdeen).

Rufener, E. Eine Bemerkung zum Zinsfussproblem. Mitt. Verein. Schweiz. Versich.-Math. 51, 211–215 (1951).

Integral Equations

✓★**Schmeidler, Werner. Integralgleichungen mit Anwendungen in Physik und Technik. I. Lineare Integralgleichungen.** Mathematik und ihre Anwendungen in Physik und Technik, Reihe A, Band 22. Akademische Verlagsgesellschaft, Leipzig, 1950. xii+611 pp. 38 DM.

This book is a comprehensive treatise on the theory of linear integral equations. Besides the classical theory of nonsingular equations, of the type for which the Fredholm theorems hold, many classes of singular equations are dis-

cussed. The principal tool used is the concrete Hilbert space of sequences (x_n) of complex numbers for which the series $\sum_{n=1}^{\infty}|x_n|^2$ is convergent.

The book opens with introductory sections containing historical remarks, the classification of linear integral equations into those of the first, second and third kinds, a brief discussion of equations with degenerate kernels, and an exposition of the basic theory of orthogonal systems of functions.

Next comes a very full discussion of linear integral equations of the first kind, the emphasis being on those whose kernel represents a bounded, but not as a rule completely continuous, operator in Hilbert space. After an exposition of the theory of linear and bilinear forms in an infinity of variables, the author discusses many special types of equations with singular kernels, including equations involving Cauchy principal values, equations associated with the Fourier transform and its generalizations, and equations whose kernels are functions of the sum, difference or product of the variables. The general theory of equations of the first kind follows; it is based on the reduction of the infinite matrix associated with the kernel to a special canonical form. Cases when this canonical form is especially simple include unitary, Hermitian and normal operators. A discussion of the characteristic values and functions of Hermitian and normal kernels appears here, a little surprisingly, instead of in the later chapter on equations of the second kind. This chapter concludes with Volterra equations of the first kind.

The chapter on equations of the second kind opens with Volterra equations; then come the remainder of the Schmidt theory for Hermitian and normal kernels, the determinant-free Fredholm theorems for Hilbert-Schmidt kernels, the Neumann series, the theory of principal kernels and functions, and a discussion of Lewin's "eigentümliche Werte". After dealing with the Fredholm determinants for continuous kernels, the author goes on to the expansion theorems for normal kernels and functions representable by them, symmetrizable kernels, and the asymptotic behavior of characteristic values and functions. Several sections on applications and special integral equations are followed by one on practical methods of solution, including approximation by degenerate kernels, iterative methods of finding the characteristic values and functions of Hermitian kernels, and the Gauss-Nyström and Enskog methods. Then come some sections on singular equations of the second kind, mainly from the point of view of the spectral decomposition of linear operators, the final section of the chapter being on the Carleman theory for unbounded Hermitian operators.

The book concludes with a chapter on equations of the third kind, and an appendix containing auxiliary matter on integration, the Hilbert space of functions of integrable square, linear and bilinear forms in an infinity of variables, and systems of linear equations in an infinity of unknowns. There is also an extremely useful index of special integral equations solved or discussed in the course of the book.

Although the book is not very systematically arranged, it will be of great value to those concerned in any way with linear integral equations, especially for the wealth of information it contains on special types of integral equations and on their applications in numerous domains of mathematical physics and engineering. The author indicates his intention of producing a further volume or volumes to cover non-linear integral equations and integro-differential equations. *F. Smithies* (Cambridge, England).

★Petrovskiĭ, I. G. Lekcii po teorii integral'nyh uravneniĭ. [Lectures on the Theory of Integral Equations]. 2d ed. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1951. 127 pp.

The author gives an exposition of the fundamental theorems of integral equations of the second kind:

$$\varphi(P)=\int_G K(P,Q)\varphi(Q)dQ+f(P),$$

where P, $Q\epsilon G$, a d-dimensional region bounded by a finite number of $(d-1)$-dimensional smooth surfaces. In chapter 1 the Fredholm theorems are stated and then proved, first for a degenerate kernel, next for continuous kernels small in absolute value, then for uniformly continuous kernels, and finally for kernels of the form $\bar{K}(P,Q)/PQ^{\alpha}$ where $\alpha<d$ and $\bar{K}$ is uniformly continuous. In chapter 2 Volterra equations are treated briefly by reduction to the last case of chapter 1. In chapter 3 real symmetric piecewise continuous kernels are treated. Here the author emphasizes the analogy with transformations of n-dimensional vectors by symmetric matrices, setting forth the analogous theorems in parallel columns. The rest of the chapter is then devoted to proving these theorems for the integral equation case. In an appendix the author indicates the necessary modifications in order to carry through the theorems of chapter 3 for real symmetric Lebesgue square integrable kernels. The book is written simply and clearly and with good motivation for the theorems.

J. V. Wehausen (Providence, R. I.).

Taldykin, A. T. On the theory of linear integral equations. Mat. Sbornik N.S. **29**(71), 281–312 (1951). (Russian)

The integral equations are of Fredholm type (*) $\varphi-N\varphi=f$ where $N\varphi=\int_a^b N(x,y)\varphi(y)dy$, integration being usually understood in the Riemann sense, and the kernel N being supposed of integrable square in x and y separately and in both together. In addition, the sets of right and left characteristic functions $\{\varphi_i\}$ and $\{\psi_i\}$, where $\varphi_i=\lambda_i N\varphi_i$, $\psi_i=\lambda_i N^*\psi_i$, with $N^*(x,y)=N(y,x)$, are assumed to form a pair of biorthogonal sets of one or other of the types discussed by the author previously [Mat. Sbornik N.S. **29**(71), 79–120 (1951); these Rev. **13**, 253]. If the sets are both normal the following assertions hold. (Ia) $\sum|\varphi_i(x)/\lambda_i|^2$, (Ib) $\sum|\psi_i(x)/\lambda_i|^2$, are convergent for all x. (II) Conversely, (Ia) implies that $\{\varphi_i\}$ are right characteristic functions of a kernel N for an integral equation in the Lebesgue sense. (III) $\sum|\lambda_k|^{-2}$ is convergent. (IV) $\sum\varphi_i(x)\varphi_i(y)/\lambda_i$ is convergent in mean with respect to x and y and with respect to x, y separately. (IV) $\sum\varphi_i(x)\psi_i(y)/\lambda_i^p$ is uniformly and absolutely convergent in both variables. (Note: though inadvertently omitted from the hypotheses, the assumption $p\geqq 3$ is necessary for the proof.) (V) If $f=Nh$, and $A^k=\int_a^b f(x)\psi_k(x)dx$, then $\sum A^k\varphi_k(x)$ is uniformly and absolutely convergent to the projection of f on the space spanned by $\{\varphi_i\}$. (VI) The series of (IV) with $p=2$ converges absolutely and uniformly in each of the variables separately. (VII) If λ is not a characteristic value (*) has the solution $\varphi=f-\lambda\sum A^k\varphi/(\lambda-\lambda_k)$. (VIII) $f=N\varphi$ has a solution if $\sum|\lambda_k A^k|^2<\infty$.

If only $\{\varphi_i\}$ is normal, (Ia), (III), and, under some further restrictions, (VII) hold. If the $\{\varphi_i\}$ are such that the biorthogonal series $\sum A^k\varphi_k(x)$ is convergent in mean for all f, then (V) and (VII) hold. $f=N\varphi$ for some φ if $\sum\lambda_i\bar{\lambda}_j A^i\bar{A}^j(\varphi_i,\varphi_j)$ is convergent. *J. L. B. Cooper.*

Elliott, Joanne. On some singular integral equations of the Cauchy type. Ann. of Math. (2) **54**, 349–370 (1951).

The integral equation (*) $\phi(z)=\pi^{-1}\int_C f(\zeta)(\zeta-z)^{-1}d\zeta$, C a curve in the ζ-plane, z on C, $\int_C$ the Cauchy principal value, is discussed with a view to finding solutions f which are in L^p as functions of arc length on C. Conditions imposed on C, parametrised by arc length as $\zeta=\zeta(s)$, are

$$(1)\qquad |\zeta(s)-\zeta(t)|\cdot|s-t|^{-1}>m>0;$$

$$(2)\qquad |\zeta''(s)|<K;$$

$$(3)\qquad \left|\zeta'(s)-\frac{\zeta(s)-\zeta(t)}{s-t}\right|<M|s|^{-1}$$

for $a<|s-t|<a|t|$, with $0<a<1$. Then, if C extends to infinity at both ends $(-\infty<s<\infty)$, $p>1$, and $\phi\epsilon L^p$, there is a unique solution $f\epsilon L^p$. If C is semi-infinite $(0\leqq s<\infty)$, $p>2$, and $s^{\frac{1}{2}}\phi(s)\epsilon L^p$, there is a unique solution such that $s^{\frac{1}{2}}f(s)\epsilon L^p$. In either case, (3) can be omitted if the assumptions on ϕ are slightly strengthened. If C is a closed curve, satisfying a modified form of (2) and (1), there is a unique L^p-solution if $\phi\epsilon L^p$ $(p>1)$. Finally, a result is proved for finite curves consisting of several pieces. The proofs use the results and methods of M. Riesz [Math. Z. **27**, 218–277 (1927)] and the complex variable method employed by T. Carleman [ibid. **15**, 111–120 (1922)]. *G. E. H. Reuter.*

Bondar', N. G. On the approximation of fundamental functions and functions of small dynamical displacements of rod systems. Akad. Nauk SSSR. Prikl. Mat. Meh. **15**, 207–226 (1951). (Russian)

The integro-differential equation governing small forced damped vibrations of rod systems (in the plane or in space) is

$$y(x,t)+\int_l G(x,s)[\ddot{y}(s,t)+\psi\dot{y}(s,t)]d\sigma(s)=\int_l G(x,s)dR(s,t),$$

where $y(x,t)$ is the displacement, $G(x,s)$ is the influence function of the system, ψ is a constant, $dR(s,t)$ and $d\sigma(s)$ are force and mass, respectively, and the integration is in the sense of "Čebyšev-Stieltjes". The solution of this integro-differential equation may be given as an infinite series which involves the eigenfunctions and eigenvalues of the integral equation

$$(*)\qquad \varphi(x)=\lambda\int_l G(x,s)\varphi(s)d\sigma(s),$$

whose kernel is symmetric and positive definite [Gantmacher and Krein, Oscillation matrices and small oscillations of mechanical systems, Moscow-Leningrad, 1941; these Rev. **3**, 242]. The author gives first two general theorems concerning the numerical approximation of the eigenfunctions and eigenvalues of (*), and then applies his results to the determination of the displacement of rod systems. Numerical applications are given. *J. B. Diaz.*

Functional Analysis, Ergodic Theory

Takenouchi, Osamu. Une démonstration directe d'un théorème de M. G. W. Mackey. Kōdai Math. Sem. Rep. **1951**, 49–50 (1951).

Let N_1 and N_2 be pseudo-norms (i.e. norms which may be zero at nonzero elements) on the same vector space X. The author shows that every linear functional on X which is bounded with respect to (N_1+N_2) is of the form f_1+f_2 where f_1 and f_2 are bounded with respect to N_1 and N_2 respectively. The reviewer [Trans. Amer. Math. Soc. **57**, 155–207 (1945), Theorem VII-5; these Rev. **6**, 274] proved an equivalent theorem by generalizing slightly certain arguments of Krein and Smulian about regularly convex sets. The author's proof is direct and elementary.

G. W. Mackey (Cambridge, Mass.).

Tamura, Takayuki. On a relation between local convexity and entire convexity. J. Sci. Gakugei Fac. Tokushima Univ. **1**, 25–30 (1950).

Call a set M in a real Banach space B locally convex if for each point x of B there is a sphere S_x centered at x such that $S_x\cap M$ is convex or empty. (Clearly this is a restriction only at boundary points of M.) Theorem. If M is locally convex and arcwise connected, then M_i, the interior of M, is convex. The proof uses the following lemmas. (I) If a and b are points such that the open segment connecting them is in M_i, then there exists a sphere $S_b'\subset S_b$ such that for each point y in $S_b'\cap M$ the open segment from a to y is in M_i. (II) If a is an interior and b a boundary point of a convex set, then the extension of the segment from a beyond b is exterior to the set. To prove the theorem, let a and $b\epsilon M_i$ and let $F(\lambda)$, $0\leqq\lambda\leqq 1$, be an arc connecting them in M. (I) asserts that the set of those λ for which the open segment $(a, F(\lambda))$ is interior to M is open. Therefore, if the segment $(a, F(1))\nsubseteq M_i$, there is a smallest λ_0 such that $(a, F(\lambda_0))\nsubseteq M_i$. If c is the boundary point of M nearest to a on the segment $(a, F(\lambda_0))$, (II) implies that some points of the segment near c are exterior to M, but all points of that segment are limits of points of the segments $(a, F(\lambda))$, $\lambda<\lambda_0$. This contradiction shows that the segment from a to b is in M_i.

Reviewer's remark. If M is locally convex, if $a\epsilon M_i$, and if E_a is the set of those $x\epsilon M$ such that the open segment $(a,x)\subseteq M_i$, then (I) shows E_a open in M and (II) shows it closed in M. Hence arcwise connectivity can be replaced by connectivity of M. *M. M. Day* (Urbana, Ill.).

Ruston, A. F. Direct products of Banach spaces and linear functional equations. Proc. London Math. Soc. (3) **1**, 327–384 (1951).

This paper continues an earlier paper [same Proc. (2) **53**, 109–124 (1951); these Rev. **13**, 138] by the same author on the generalization of the Fredholm equation $x=y+\lambda Kx$ where x and y are elements of a Banach space $\mathfrak{B}$ and K is a linear operation on $\mathfrak{B}$ of the trace class. The author studies the product of n Banach spaces and generalizes the work of Schatten on two factors to this case. The obvious notions appear, equivalence, norms, cross norms, associate norms, norm uniformity, the greatest cross norm γ and least cross norm λ. A method of assigning a norm to a tensor product $\mathfrak{B}_1\odot\mathfrak{B}_2\odot\cdots\odot\mathfrak{B}_n$ is called a "clorm". γ and λ are examples of clorms as well as the self-conjugate norm of Schatten's first paper [Trans. Amer. Math. Soc. **53**, 195–217 (1943); these Rev. **4**, 161]. This latter notion applies to self-conjugate spaces and is generalized here to the case of n factors. The compounding of operators on $\mathfrak{B}$ which yield operators on the product of n Banach spaces is described and in particular a trace class of operators from the n-factor γ-normed space to the n-factor λ-normed space is specified with a trace norm and trace operation from the trace class of n factors to that of $n-1$ factors. These notions of the cross product of n Banach spaces and trace operations yield an abstract equivalent of the multi-integrals used in the regular Fredholm theory. The latter can then be completely

generalized: there is a numerically valued function $D_0(\lambda)$ which is such that if $D_0(\lambda)\neq 0$, the solution x of the above problem is unique while if $D_0(\lambda)=0$, the previously constructed equivalents of the multi-integrals can be used to specify the set of x's for which $(1-\lambda K)x=0$ and the y's for which a solution can be obtained. The author then obtains a complete solution of the above Fredholm problem.

F. J. Murray (New York, N. Y.).

Kantorovič, L. V. Some further applications of the principle of majorants. Doklady Akad. Nauk SSSR (N.S.) 80, 849–852 (1951). (Russian)

This is a continuation of an earlier paper [same Doklady 76, 17–20 (1951); these Rev. 12, 835] and applies the same methods to discuss the uniqueness of solution of equations of the form $U(x)=x$ and the domain of attraction of a solution x^* of $U(x)=x$ [i.e. the points x' such that $x_n\to x^*$, $x_0=x'$, $x_{n+1}=U(x_n)$]. Here $x\varepsilon X$ where X is a linear space whose elements are "normed" by the elements of another partially ordered linear space Z. Applications are made to Newton's method [cf. the cited review and reference cited there]. Transformations depending upon a parameter are also considered.

J. V. Wehausen (Providence, R. I.).

***Aronszajn, N. Approximation methods for eigenvalues of completely continuous symmetric operators.** Proceedings of the Symposium on Spectral Theory and Differential Problems, pp. 179–202. Oklahoma Agricultural and Mechanical College, Stillwater, Okla., 1951. $3.00.

Let D be a domain in a finite-dimensional Euclidean space. Let K be the class of $2r$ times differentiable complex valued functions u defined in D, and A, B linear differential operators defined in the class K; A is of order $2t$, B of smaller order. If the quadratic form (1) $\mathfrak{A}(u,u)=\int_D A(u)\bar{u}dt$ is positive definite while (2) $\mathfrak{B}(u,u)=\int_D B(u)\bar{u}dt$ is real, then eigenvalue problems concerning $A(u)=\mu B$ (with suitable linear homogeneous boundary conditions) may in many cases be treated as follows [cf. K. O. Friedrichs, Math. Ann. 109, 465–487 (1934); N. Aronszajn, Studies in Eigenvalue Problems, I, Oklahoma A. and M. College, Stillwater, 1949]: define in K a norm by setting (3) $\|u\|^2=\mathfrak{A}(u,u)$, and let $\bar{K}$ be the Hilbert space obtained by completing K (with respect to the norm (3)). $\mathfrak{B}(u,u)$ is (under suitable assumptions) densely defined and bounded, and can therefore be extended uniquely to $\bar{K}$. Then by a theorem of Friedrichs there exists a uniquely determined linear bounded operator L on K such that (4) $\mathfrak{B}(u,v)=\mathfrak{A}(L(u),v)$, where in an obvious manner $\mathfrak{A}(z,v)$, $\mathfrak{B}(u,v)$ denote the bilinear forms belonging to the quadratic forms (1), (2) resp. The operator L is completely continuous if the original differential eigenvalue problem has a discrete spectrum, and the eigenvalues μ_i of the latter problem are the reciprocals of the eigenvalues λ_i of L. Moreover, if L is positive definite the λ_i can also be defined as the maximum of the quotient (5) $\mathfrak{B}(u,u)/\mathfrak{A}(u,u)$ under certain well known linear side conditions (Courant's minimum-maximum principle). Finally, the eigenelements of L or the solutions of the variational problem (5) give the solution of the original differential eigenvalue problem (if the latter was "properly stated").

The present paper deals with this method in abstracto; i.e., without reference to a differentiable eigenvalue problem: there is given a vector space V over the complex numbers. The notions of linear, bilinear and quadratic forms are explained. If $\mathfrak{A}(u,v)$ is a bilinear form for which $\mathfrak{A}(u,u)$ is positive definite, V becomes a normed space $V_{\mathfrak{A}}$ by introducing a norm by (3). As above, $V_{\mathfrak{A}}$ can be completed to a Hilbert space $\bar{V}_{\mathfrak{A}}$. The extension of $\mathfrak{A}$ to $\bar{V}_{\mathfrak{A}}$ will be denoted by $\mathfrak{A}$. If $\mathfrak{B}(u,v)$ is another bilinear form defined in V the notion of boundedness of $\mathfrak{B}$ with respect to $\mathfrak{A}$ is introduced. If $\mathfrak{B}$ is bounded with respect to $\mathfrak{A}$, there exists a bounded linear operator L with domain V and possessing an adjoint L^* such that

$$\mathfrak{B}(u,v)=\mathfrak{A}(Lu,v)=\mathfrak{A}(u,L^*v)$$

(cf. 4). $\mathfrak{B}$ is called completely continuous relative to $\mathfrak{A}$ if L is completely continuous. If in addition $\mathfrak{B}$ is real, L turns out to be selfadjoint ($L=L^*$), and the ordinary spectral theorem for selfadjoint completely continuous linear operators can be applied to L (extended to $\bar{V}_{\mathfrak{A}}$). The eigenelements of L don't have to be in V. However the positive and negative eigenvalues can be defined without the extension to $\bar{V}_{\mathfrak{A}}$ by variational problems of the minimum-maximum (or rather inf-sup) type connected with the quotient (5).

For these variational problems approximation methods (including the generalized Rayleigh-Ritz and Weinstein methods) are given which are in part analogous to those used by the author for differential eigenvalue problems in the notes cited above. Indications about the convergence proofs for the approximations conclude the paper.

E. H. Rothe (Ann Arbor, Mich.).

Phillips, R. S. Spectral theory for semi-groups of linear operators. Trans. Amer. Math. Soc. 71, 393–415 (1951).

The structure of semi-groups of operators on a Banach space is investigated by (a) considering a naturally engendered commutative normed ring with unit and (b) discussing the maximal ideals of this ring via those of an easily associated function ring with convolution.

The semi-group $T(t)$ is called of type H if:

(1) $\qquad T(t_1+t_2)=T(t_1)T(t_2),\quad T(0)=I;$

(2) $T(t)x$ is strongly measurable; (3) $\int_0^1\|T(t)\|dt<\infty$; (4) $\lim_{\tau\to 0}\tau^{-1}\int_0^\tau T(t)xdt=x$ for all x. In terms of

$$\omega(t)=\log\|T(t)\|,$$

the Banach algebra $S(\omega)$ of additive set functions $\alpha(\sigma)$ on the σ-field of Baire sets of $[0,\infty)$ for which $\int_0^\infty e^{\omega(t)}|d\alpha|<\infty$ is defined with convolution for multiplication. If R_1 is the strong closure of polynomials in $T(t)$, $R(\lambda, T(t))=$resolvent of $T(t)$ ($\lambda\varepsilon\rho(T(t))=$resolvent set of $T(t)$, $t\geqq 0$, and $R(\lambda, A)$ ($\lambda\varepsilon\rho(A)$, A the infinitesimal generator of the semi-group), then R is defined to be the strong closure of the polynomials of $\{R_1\cup$ inverses of elements in $R_1\}$. In particular, R contains $\theta(\alpha)$ defined by $\theta(\alpha)x=\int_0^\infty T(t)xd\alpha$, $\alpha\varepsilon S(\omega)$, and thus $S(\omega)$ is continuously homomorphically mapped into R.

If M is the set of maximal ideals of $S(\omega)$, M' that of R, m and m' maximal ideals in M and M', μ_m and $\mu_{m'}$ the associated multiplicative functionals, then $\mu_m(\alpha)=\mu_{m'}(\theta(\alpha))$ defines μ_m in M when $\mu_{m'}$ is given and thus M' is mapped onto M. In M the sets $W=$maximali deals not containing the set of absolutely continuous elements of $S(\omega)$, $Z=$maximal ideals of all α for which $\alpha(0)=0$, and $\mathfrak{U}=M-(W\cup Z)$ are defined as well as their inverse images W', Z', $\mathfrak{U}'$ in M'. The set W is homeomorphic with the set of complex numbers $\{\lambda\,|\,R(\lambda)\leqq\omega_0=\inf_{t>0}\omega(t)/t\}$ and if $m\varepsilon W$, and μ_m is the associated multiplicative functional, then $\mu_m(\alpha)=\int_0^\infty e^{\lambda t}d\alpha$ where λ is the image of μ_m. Thus for each $m'\varepsilon M'$, there is a complex number $a(m')$ such that $T(t)(m')=e^{a(m')t}$ and

$$\theta(\alpha)(m')=\int_0^\infty e^{a(m')t}d\alpha$$

for $m'\varepsilon W'$, and 0 otherwise, for α absolutely continuous.

If the semi-group is of type H, then the following are equivalent: (a) $T(t)=\exp(At)$ where A is bounded; (b) $M'=W'$; (c) $T(t)(m')=e^{a(m')t}$. If $T(t)$ is continuous in the uniform topology at $t=0$, then the spectrum of

$$\theta(\alpha)=\Sigma(\theta(\alpha))=f_\alpha(\Sigma(A))$$

for all $\alpha\epsilon S(\omega)$, where $f_\alpha(\lambda)=\int_0^\infty e^{\lambda t}d\alpha$. If the semi-group is of type H and the infinitesimal generator A is unbounded, then $\Sigma(\theta(\alpha))\supset f_\alpha(\Sigma(A))$ for $\alpha\epsilon S(\omega)$ and $\Sigma(\theta(\alpha))=f_\alpha(\Sigma(A))\cup 0$ for α absolutely continuous. The case in which $T(t)$ is continuous in the uniform topology for $t\geqq c>0$, and the semi-group is of type H is also treated, and analytical semi-groups are discussed.

The concluding section gives a proof of a generalization of Stone's theorem on groups of unitary operators, viz.: If $U(g)$ is a strongly continuous unitary representation of a locally compact abelian group G, then there is a resolution of the identity $F(\sigma)$ on the Borel sets of the character group G^* for which $U(g)=\int_{G^*}\chi(g)dF(\sigma)$. *B. Gelbaum.*

Grinblyum, M. M. Spectral measure. Doklady Akad. Nauk SSSR (N.S.) **81**, 345–348 (1951). (Russian)

In this (posthumous) note the author continues his investigation of functions defined on points, intervals, or sets for which the values lie in the set of projections of a given reflexive Banach space B. Starting from such a function $P(\Delta)$ defined for intervals Δ contained in the real line I, with the properties assumed in the preceding papers [same Doklady **70**, 749–752, 941–944 (1950); these Rev. **11**, 525], $P(M)$ is defined for Borel sets M by transfinite induction on the successive classes O_λ and F_λ of Borel subsets of I, after first being defined on open sets $O=\bigcup_{i<\infty}\Delta_i$ by setting $P(O)=$ least projection whose range covers the ranges of all $P(\Delta_i)$. A Borel set A_0 is said to be of P-measure zero if $P(A_0)$ is the zero projection in B. The spectral measure P is completed by extending it to the Borel field of all sets $M=A\cup N$, A a Borel set, N a subset of a Borel set of P-measure zero, by setting $P(M)=P(A)$. It is asserted that P is completely additive on this Borel field, and that if M_n are sets in the field such that $M_i\supseteq M_{i+1}$ for each i, and if $M=\bigcap_{i<\infty}M_i$, then (1) the range of $P(M)=\bigcap_i$ {range of $P(M_i)$}, and (2) $P(M_i)\rightarrow P(M)$ in the strong topology in the space of linear operators from B into B.

M. M. Day (Urbana, Ill.).

Walters, Stanley S. Remarks on the space H^p. Pacific J. Math. **1**, 455–471 (1951).

This note concerns an integral representation of linear functionals on H^p, the space of functions analytic inside the unit circle for which

$$\|f\|=\sup_{0\leqq r<1}\left(\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^p d\theta\right)^{1/p}$$

is finite. The results are these: If γ is a linear functional on H^p, $0<p<1$, then there exists a unique G analytic in $|z|<1$ and continuous in $|z|\leqq 1$ such that for each $f\epsilon H^p$

$$\gamma(f)=\lim_{r\rightarrow 1}\frac{1}{2\pi}\int_0^{2\pi}f(\rho e^{i\theta})G\left(\frac{r}{\rho}e^{-i\theta}\right)d\theta,\quad r<\rho<1.$$

Moreover, (a) if $0<p<1/n$, $n=2, 3, \cdots$, then the $(n-1)$st derivative of G with respect to z is continuous on $|z|\leqq 1$; (b) if $0<p<1/2n$, then $G(e^{it})$ has a continuous nth derivative with respect to t; and (c) if $0<p<\frac{1}{2}$, then the power series for G converges absolutely on $|z|<1$. These results are related to some recent work of A. E. Taylor [Studia Math. **11**, 145–170 (1950); **12**, 25–50 (1951); these Rev. **13**, 45, 252]. *M. M. Day* (Urbana, Ill.).

Lorentz, G. G. On the theory of spaces Λ. Pacific J. Math. **1**, 411–429 (1951).

The author extends his earlier investigation of the Banach spaces $\Lambda(\alpha)$ and $M(\alpha)$ [Ann. of Math. (2) **51**, 37–55 (1950); these Rev. **11**, 442]. Let $[0, l]$ be a finite interval; let $\phi(x)$ be a fixed function, integrable and non-negative on $(0, l)$, and not identically zero; let the function $f(x)$ be measurable on the interval $(0, l)$; and let $f^*(x)$ denote the decreasing rearrangement of $|f(x)|$, that is, the decreasing function on $(0, l)$ which is equimeasurable with $|f(x)|$. Then $f(x)\epsilon\Lambda(\phi, p)$ provided the quantity

$$(1)\qquad \|f\|=\left\{\int_0^l\phi(x)f^*(x)^p dx\right\}^{1/p}$$

is finite. The space $\Lambda(\phi, p)$ with the norm given by (1) is a Banach space if and only if the function $\phi(x)$ is equivalent (in the sense of equality almost everywhere) to a decreasing function; if $p>1$, the space is reflexive; it is not necessarily uniformly convex.

If C is a class of non-negative integrable functions $\phi(x)$, the author defines the space $X(C, p)$ to consist of all functions $f(x)$ for which the norm

$$\|f\|=\sup_{\phi\epsilon C}\left\{\int_0^l\phi|f|^p dx\right\}^{1/p}$$

is finite. He proves theorems concerning Hardy-Littlewood majorants; the following is the simplest example of these: If $f\epsilon\Lambda(\phi, p)$ with $p>1$, and if

$$\theta(x, f)=\sup_{0\leqq y\leqq l}\frac{1}{y-x}\int_x^y|f(t)|\,dt,$$

then $\theta(x, f)\epsilon\Lambda(\phi, p)$. The author also applies his results to integral transformations, and he characterizes the sequences $\{\mu_n\}$ for which the equation $\mu_n=\int_0^l x^n f(x)dx$ has a solution in the space $\Lambda(\phi, p)$. *G. Piranian* (Ann Arbor, Mich.).

Haplanov, M. G. Linear transformations of analytic spaces. Doklady Akad. Nauk SSSR (N.S.) **80**, 21–24 (1951). (Russian)

The author obtains the following results on linear transformations $y=T(x)$ of an analytic space, where $M=(a_{ij})$ is the matrix of the transformation (for definitions see the review of an earlier paper [same Doklady **79**, 929–932 (1951); these Rev. **13**, 252]). (1) T maps A_1 into itself (or $\bar{A}_1$ into A_1, if T operates only on $\bar{A}_1$) if and only if for every $q>1$ there exists a sequence $\{c_n\}$ of positive numbers with $\lim[c_n]^{1/n}<1$ such that $|a_{jn}|\leqq c_j q^n$ for $j, n=1, 2, \cdots$. (2) T maps A_1 into itself (into $\bar{A}_1$) if and only if the columns of M are dominated by the coordinates of a single point c in A_1 ($\bar{A}_1$) and there exists q, $0<q<1$, and $\nu>0$ such that $|a_{jn}|\leqq q^n$ for $n>\nu\cdot j$, $n, j=1, 2, \cdots$. (3) T maps l^p $(1\leqq p\leqq\infty)$ into A ($\bar{A}$) if and only if $\lim(\|a^{(j)}\|)^{1/j}\leqq 1$ (<1), where $\|a^{(j)}\|$ is the norm of the jth row of M in the metric of $l^{(p/(p-1))}$. He remarks that these theorems can be used to decide whether or not a given sequence of functions forms a basis for the space of functions analytic in $|z|<R$.

B. Crabtree (Durham, N. H.).

Bondarenko, P. S. On uniqueness for infinite systems of linear equations. Mat. Sbornik N.S. **29**(71), 403–418 (1951). (Russian)

Various theorems on uniqueness are obtained for the system (I) $x_i=\sum_{k=1}^{\infty}a_{ik}x_k+b_i$ $(i=1, 2, \cdots)$, usually in relation to a majorant system (II) $X_i=\sum_{k=1}^{\infty}A_{ik}X_k+B_i$ $(i=1, 2, \cdots)$, defined as any system for which $|a_{ik}|\leqq A_{ik}$ and $|b_i|\leqq B_i$. A principal tool is the method of successive approximations, whereby, starting with initial values $\bar{x}_i^{(0)}$, one defines a sequence of approximations by induction: $\bar{x}_i^{(n)}=\sum_{k=1}^{\infty}a_{ik}\bar{x}_k^{(n-1)}+b_i$ $(i=1, 2, \cdots; n=1, 2, \cdots)$. If the initial values $\bar{x}_i^{(0)}$ are all zero, the corresponding solution x_i of (I), if it exists, is called the principal solution. (Correspondingly for system (II).)

Some results found in Kantorovič and Krylov [Approximate methods of higher analysis, 3d ed., Moscow-Leningrad, 1950, pp. 31–35; these Rev. **13**, 77] are stated for reference. Among the theorems proved the following are typical: (i) If system (II) satisfies condition (c) $\sum_{k=i+1}^{\infty}A_{ik}\leqq MB_i$ (M=constant), then (II) cannot have a non-negative solution different from the principal solution; and if (c) holds and (II) has a solution bounded below by a positive number, then system (I) has a unique bounded solution. (ii) If system (II) is regular (i.e., $\sum_{k=1}^{\infty}A_{ik}<1$, $i=1, 2, \cdots$), satisfies condition (c), and has a non-negative solution, then system (I) has a unique bounded solution, and this solution can be obtained by the method of successive approximations, starting with an arbitrary bounded set of initial values $\bar{x}_i^{(0)}$. (iii) A regular system cannot have more than one solution tending to zero.

An example in the theory of equilibrium of elastic square plates is given in which the uniqueness of a bounded solution is established by means of results of this article.

I. M. Sheffer (State College, Pa.).

★**Lorch, E. R. The spectral theorem and classical analysis.** Proceedings of the Symposium on Spectral Theory and Differential Problems, pp. 259–265. Oklahoma Agricultural and Mechanical College, Stillwater, Okla., 1951. $3.00.

L'auteur esquisse une démonstration nouvelle de la formule bien connue $H=\int_{-\infty}^{+\infty}\lambda dE(\lambda)$, H opérateur self-adjoint quelconque (non nécessairement borné). La démonstration est basée sur l'usage des opérateurs

$$K_{\lambda\mu}(m, n)=\frac{1}{2\pi i}\int_C(z-\lambda)^m(\mu-z)^n(z-H)^{-1}dz$$

où m, n sont des entiers positifs et où C est un cercle coupant l'axe réel en λ et μ (ces intégrales convergent dans la topologie uniforme si l'on suppose que le spectre ponctuel de H est vide), et il se trouve que les variétés spectrales cherchées $\mathfrak{M}(\lambda, \mu)$ sont les adhérences des domaines des valeurs des opérateurs $K_{\lambda\mu}(m, n)$ (quels que soient m et n). L'auteur termine par quelques considérations sur les transformations uniformément bornées ($\sup_n \|T^n\|<+\infty$).

R. Godement (Nancy).

Régnier, André. Enveloppes d'opérateurs hermitiens bornés. C. R. Acad. Sci. Paris **232**, 675–677 (1951).

Régnier, André. Enveloppes d'opérateurs hermitiens bornés. C. R. Acad. Sci. Paris **232**, 920–921 (1951).

The author proves the known fact that, if G is a set of hermitian operators which coincides with its commutor, then G is a vector σ-lattice. *L. Nachbin.*

★**Rellich, Franz. Störungstheorie der Spektralzerlegung.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 1, pp. 606–613. Amer. Math. Soc., Providence, R. I., 1952.

Let $A(e)$ be a family of operators self-adjoint on a fixed domain $\mathfrak{D}$, dense in the Hilbert space $\mathfrak{H}$. Perturbation theory seeks to determine how the spectrum, eigenvalues, and eigenvectors vary with real e near zero. Such questions arise in frequently used techniques in applied mathematics, particularly quantum physics, though in application the needed conditions for validity are often not available. Rellich's paper summarizes the results obtained in the last fifteen years by himself and others, Friederichs and Nagy in particular, as noted in the bibliography.

Suppose $A(e)$ is regular on $\mathfrak{D}$. Specifically for $u\varepsilon\mathfrak{D}$ $A(e)u=A(0)u+\sum_{p=1}^{\infty}e^pA_pu$ convergent with

$$\|A_pu\|\leqq k(M)^p(\|u\|+\|A(0)u\|),$$

where A_p is symmetric on $\mathfrak{D}$. Now if the interval $a<\lambda<b$ contains only point spectra of $A(0)$ and has a finite-dimensional manifold of eigenvectors, then Rellich and Nagy show the eigenvalues of $A(e)$ in this interval are analytic in e and the eigenvectors regular, $E_{b-h}(e)-E_{a+h}(e)$ being regular for small $h>0$. The result is no longer true if this finiteness condition fails, as Rellich illustrates by an example in which an eigenvalue collapses into a point of the continuous spectrum of $A(0)$. Some results of Friederichs for the continuous spectrum, naturally less complete, are noted.

Suppose $A(e)$ is no longer regular in e, but merely continuous. Specifically let $A^{(0)}$ and $A^{(n)}$ all be essentially self-adjoint over a fixed dense $\mathfrak{D}$ such that $\|A^{(n)}u-A^{(0)}u\|\to 0$ as $n\to\infty$ for $u\varepsilon\mathfrak{D}$. Then $E_\lambda^{(n)}\to E_\lambda^{(0)}$ in the strong topology if λ is not in the point spectrum of $A^{(0)}$. If

$$\|A^{(n)}u-A^{(0)}u\|\leqq\rho_n(\|u\|+\|A^{(0)}u\|)$$

over $u\varepsilon\mathfrak{D}$ with $\rho_n\to 0$, and if $(\lambda-h, \lambda+h)$ contains no points in the spectrum of $A^{(0)}$ for some $h>0$, then $E_\lambda^{(n)}\to E_\lambda^{(0)}$ in the uniform topology as well. *F. H. Brownell.*

Heinz, Erhard. Beiträge zur Störungstheorie der Spektralzerlegung. Math. Ann. **123**, 415–438 (1951).

This paper cleans up a few details in the results outlined in the preceding review. For regular perturbation the author proves $E_\lambda(e)$ regular. This is not implied by regularity of $E_\lambda(e)-E_\mu(e)$, proved by Nagy if the spectrum of $A^{(0)}$ is unbounded at one end. More important, for merely continuous perturbation Rellich's proof of $E_\lambda^{(n)}\to E_\lambda^{(0)}$ in the uniform case above is not complete if $A^{(n)}$ and $A^{(0)}$ are allowed unbounded. The author gives a new complete proof of this result. *F. H. Brownell* (Seattle, Wash.).

Dixmier, J. Sur la réduction des anneaux d'opérateurs. Ann. Sci. École Norm. Sup. (3) **68**, 185–202 (1951).

This paper is a continuation of the author's studies on rings of operators [same Ann. **66**, 209–261 (1949); these Rev. **11**, 370]. It is shown that an arbitrary ring of operators has a unique decomposition into four parts. In the terminology of the reviewer [Ann. of Math. (2) **53**, 235–249 (1951); these Rev. **13**, 48] these are the parts of Types I, II_1, II (infinite), and III. A complete structure theory is given for a ring of type I: it is a unique direct sum of homogeneous parts, one for each cardinal number, and a homogeneous ring is fully determined by a certain commutative ring. (Similar results in the Type I case are given by Segal in the paper reviewed below.) The author concludes by

establishing the connection with von Neumann's reduction theory [Ann. of Math. **50**, 401–485 (1949); these Rev. **10**, 548]. It turns out that a ring of a given type splits into factors of that type almost everywhere, except possibly in the case of Type III. *I. Kaplansky* (Chicago, Ill.).

Segal, I. E. Decompositions of operator algebras. I. Mem. Amer. Math. Soc., no. 9, 67 pp. (1951). **$1.80.**

Le but de ce mémoire est apparemment de simplifier les (ou plutôt: certains des) résultats dûs à J. von Neumann [Ann. of Math. (2) **50**, 401–485 (1949); ces Rev. **10**, 548] en employant des méthodes différentes. Le Théorème 1 consiste, étant donnés un espace de Hilbert $\mathfrak{H}$, une algèbre autoadjointe $\mathbf{A}$ dans $\mathfrak{H}$, un anneau commutatif $\mathbf{C}\subset\mathbf{A}'$ et un vecteur $\mathbf{a}\in\mathfrak{H}$, à montrer que la forme linéaire positive $A\rightarrow(A\mathbf{a}, \mathbf{a})$ sur $\mathbf{A}$ peut être décomposée de façon canonique en une "somme continue" de formes positives associées aux points du spectre de $\mathbf{C}$ (lequel est un espace de Stone-Kakutani); la démonstration comprenant 8 pages, il peut être utile de référer à une note d'Adel'son-Vel'skiĭ [Doklady Akad. Nauk SSSR (N.S.) **67**, 957–959 (1949); ces Rev. **11**, 115] et aux pages 102–103 d'un article du rapporteur [Ann. of Math. **53**, 68–124 (1951); ces Rev. **12**, 421], d'autant plus que l'auteur ne cite pas ces articles. Ceci fait, l'auteur étudie les sommes continues d'espaces de Hilbert; il en donne une définition (en fait, deux définitions: une "forte" et une "faible") analogue à celle de von Neumann, à ceci près qu'on considère des mesures sur des ensembles quelconques. Le Théorème 2 de l'auteur dit que la décomposition du Théorème 1 correspond à une décomposition du système $\{\mathfrak{H}, \mathbf{A}, \mathbf{C}\}$ en somme continue (au moins si $\mathbf{a}$ est "générateur" relativement à $\mathbf{A}$); ceci aussi est démontré par le rapporteur dans l'article cité ci-dessus. L'auteur démontre ensuite que, si $\mathbf{C}$ est maximale dans $\mathbf{A}'$, alors presque toutes les formes positives construites au Théorème 1 sont élémentaires; la méthode de l'auteur est essentiellement celle des "choix mesurables" de von Neumann; l'auteur dit qu'il n'utilise pas les ensembles analytiques; c'est rigoureusement vrai—mais ils sont remplacés par les ensembles boréliens et un lemme sur les inverses des applications continues d'un espace compact séparables (lemme dû essentiellement à Federer et Morse [Bull. Amer. Math. Soc. **49**, 270–277 (1943); ces Rev. **4**, 213], référence pas donnée par l'auteur); notons que la méthode du rapporteur évite complètement l'utilisation de tels théorèmes (qui ne sont indispensables que pour démontrer un résultat nettement plus fort, à savoir le lemme 7 de l'article cité de von Neumann). L'auteur donne ensuite des applications: 1) décomposition des représentations unitaires d'un groupe séparable; 2) décomposition d'une mesure invariante en parties ergodiques; 3) décomposition de la "double représentation régulière" d'un groupe séparable unimodulaire en composantes irréductibles; les deux premières applications sont déjà traitées par le rapporteur dans l'article cité ci-dessus; la troisième est ni plus ni moins évidente que la première étant donné qu'une double représentation de G est aussi une représentation de $G\times G$. Observons encore que là aussi l'auteur définit les doubles représentations sans aucune référence, bien que le rapporteur en ait parlé depuis longtemps [C. R. Acad. Sci. Paris **229**, 967–969 (1949); ces Rev. **11**, 325]. Pour terminer, l'auteur montre comment l'on peut passer des décompositions où l'espace de base Γ est le spectre d'un anneau faiblement fermé à des décompositions où l'espace de base est plus petit; ceci aussi est exposé par le rapporteur.

R. Godement (Nancy).

Segal, I. E. Decompositions of operator algebras. II. Multiplicity theory. Mem. Amer. Math. Soc., no. 9, 66 pp. (1951). **$1.80.**

La première partie de cet article traite des invariants unitaires des algèbres autoadjointes commutatives. Tout d'abord l'auteur montre que si $\mathbf{A}$ est une algèbre autoadjointe commutative maximale dans un espace de Hilbert $\mathfrak{H}$ (non nécessairement séparable) alors $\mathfrak{H}$ est isomorphe à un espace $L^2(M, \mu)$, de telle sorte que $\mathbf{A}$ devienne par cet isomorphisme l'anneau de tous les opérateurs de multiplication par les fonctions mesurables et bornées sur M—résultat évident si l'on utilise la théorie spectrale dûe au rapporteur [Ann. of Math. (2) **53**, 68–124 (1951); ces Rev. **12**, 421]. Si maintenant $\mathbf{A}$ est commutative et faiblement fermée mais non nécessairement maximale, alors on peut pour tout cardinal n former un sous-espace fermé $\mathfrak{H}_n$ de $\mathfrak{H}$, invariant par $\mathbf{A}$, de telle sorte que 1) $\mathfrak{H}$ soit la somme directe des $\mathfrak{H}_n$; 2) le système $\{\mathfrak{H}_n, \mathbf{A}\}$ soit isomorphe à n "copies" d'un système $\{\mathfrak{H}', \mathbf{B}\}$ où $\mathbf{B}$ est maximale dans $\mathfrak{H}'$. L'auteur déduit de là les théorèmes connus sur les invariants unitaires. Note du rapporteur: dans le cas d'un espace $\mathfrak{H}$ séparable, la théorie des invariants unitaires peut se présenter d'une façon beaucoup plus imagée; soit en effet $\mathbf{A}$ une algèbre autoadjointe commutative dans $\mathfrak{H}$, que nous supposerons uniformément fermée seulement; soit Z le "spectre" de $\mathbf{A}$, compact si $1\in\mathbf{A}$ (ce que nous supposerons pour simplifier), de sorte que tout $T\in\mathbf{A}$ correspond à une fonction continue $\hat{T}(\zeta)$ sur Z, et réciproquement; il est alors facile de voir que $\mathfrak{H}$ se décompose en une somme continue (plus exactement, mesurable) d'espaces de Hilbert séparables $\mathfrak{H}(\zeta)$ par rapport à une certaine mesure positive $dm(\zeta)$ (unique à une équivalence près) de telle sorte que, dans cette décomposition, la ζ-composante de $T\in\mathbf{A}$ soit le scalaire $\hat{T}(\zeta)$; les sous-espaces $\mathfrak{H}_n$ sont alors formés des $\mathbf{x}\in\mathfrak{H}$ tels que $\mathbf{x}(\zeta)=0$ dès que $\dim \mathfrak{H}(\zeta)\neq n$; et on obtient un système complet d'invariants unitaires en considérant: 1) l'espace compact Z; 2) la classe de la mesure $dm(\zeta)$; 3) la fonction mesurable $\dim \mathfrak{H}(\zeta)$.

La seconde partie étend certains des résultats précédents aux anneaux (faiblement fermés) non commutatifs; en particulier l'auteur généralise la notion de "multiplicité uniforme n" et étend au cas général la décomposition de l'espace en sous-espaces $\mathfrak{H}_n$ où $\mathbf{A}$ est de multiplicité uniforme égale à n. Le cas "multiplicité uniforme 1" se produit si et seulement si $\mathbf{A}'$ est commutative (i.e. si $\mathbf{A}'$ est le centre de $\mathbf{A}$); si n est fini, le système $\{\mathfrak{H}, \mathbf{A}\}$ est de multiplicité uniforme n si et seulement s'il est la somme directe de n copies d'un système de multiplicité uniforme 1 (autrement dit, si $\mathbf{A}'$ est de rang n sur son centre). Le cas n infini n'est étudié que pour les anneaux de type (I) (les autres présentent fort probablement, comme le dit l'auteur, un comportement pathologique), et l'on obtient dans ce cas des résultats qui généralisent complètement ceux qu'on a trouvés pour n fini, et qui du reste caractérisent les anneaux de type (I). Naturellement, tout ceci s'interprète facilement dans le cas séparable, où l'on peut utiliser la décomposition de $\mathbf{A}$ en somme directe continue de facteurs. *R. Godement.*

Dixmier, Jacques. Algèbres quasi unitaires. C. R. Acad. Sci. Paris **233**, 837–839 (1951).

It is announced that every bounded linear operator on the space $L_2(G)$ of square-integrable functions over the locally compact group G (relative to Haar measure) that commutes with all left translations is in the weak closure of the ring generated by the right translations. Previously this was known only for unimodular groups [Segal, Ann. of

Math. (2) **51**, 293–298 (1950); these Rev. **12**, 157; also (independently by essentially the same method) Godement, J. Math. Pures Appl. (9) **30**, 1–110 (1951); these Rev. **13**, 12]. More generally, the analogous result is valid in any quasiunitary algebra, where this is defined as a complex *-algebra A which is also a pre-Hilbert space on whose completion H there is given a positive invertible self-adjoint operator J such that for x, y, z in A: (1) $(x, y)=(y^*, x^*)$; (2) $(xy, z)=(y, Jx^*\cdot z)$; (3) $y\to xy$ is a continuous map (x fixed); (4) A^2 is dense in A; (5) J and J^2 are the closures of their restrictions to A^2. That is, if, respectively, L_x and R_x are for x in A the bounded extensions to H of the operations $y\to xy$ and $y\to yx$ ($y\epsilon A$), and if L and R are the weak closures of the sets of all L_x and R_x ($x\epsilon A$), then $L'=R$. Moreover J commutes with every unitary operator that commutes with all the elements of L and R. *I. E. Segal* (Chicago, Ill.).

Pallu de la Barrière, Robert. Algèbres unitaires et espaces de Ambrose. C. R. Acad. Sci. Paris **233**, 997–999 (1951).

The author defines an Ambrose space to be a Hilbert space equipped with a partially defined multiplication and a notion of adjoint which satisfy certain axioms formally weaker than those given by Ambrose [Trans. Amer. Math. Soc. **65**, 27–48 (1949); these Rev. **10**, 429] in defining the notion of H-system. The main result of this note is that every Ambrose space is in fact an H-system. It follows that a unitary algebra in the sense of Godement (that is a *-algebra with an inner product such that $(vu, w)=(u, v^*w)$, $(u, v)=(v^*, u^*)$, $v\to vu$ is continuous in v for each u, and the products uv are dense) is always imbeddable in a unique smallest H-system. *G. W. Mackey* (Cambridge, Mass.).

Koopman, B. O. A probabilistic generalization of matric Banach algebras. Proc. Amer. Math. Soc. **2**, 404–413 (1951).

The paper shows that a particular representation of a Banach algebra of bounded linear operators on a Banach space yields a Banach algebra. The proof can be simplified by explicitly making use of the fact that the elements under consideration can represent operators and then applying general theorems on operators. For example, the proof of associativity of the representing elements (generalized transition probabilities) under "multiplication" can be made by appealing to the associativity of operators under multiplication rather than by the multiple integral calculation on pp. 411–412. That the elements under consideration can be considered as representing operators is well-known and, in effect, has been pointed out by Yosida and Kakutani [Ann. of Math. (2) **42**, 188–228 (1941); these Rev. **2**, 230]. *S. Sherman* (Sherman Oaks, Calif.).

Schwartz, Laurent. Analyse et synthèse harmoniques dans les espaces de distributions. Canadian J. Math. **3**, 503–512 (1951).

This is a survey of results, many of which are definitive, about weakly closed ideals (or invariant subspaces) in rings of functions (or modules over such rings), multiplication being defined either by pointwise multiplication or convolution, the domain of the functions being essentially euclidean n-space, and a key question being whether every such ideal is the intersection of the maximal, or at least primary, ideals that contain it. The results about convolution rings are for the most part obtained from theorems about pointwise multiplication rings by taking Fourier transforms. A typical new result is: If A is the ring of all functions of class C^m on euclidean n-space ($m=0, 1, \cdots$, or ∞) and A' its dual, then every weakly closed submodule of A' (under the action via multiplication of the ring A) is generated by those one-point-support distributions which it contains. *I. E. Segal* (Princeton, N. J.).

Maeda, Fumitomo. Embedding theorem of continuous regular rings. J. Sci. Hiroshima Univ. Ser. A. **14**, 1–7 (1949).

T. Iwamura [Jap. J. Math. **19**, 57–71 (1944); these Rev. **8**, 35] has shown that if L is a reducible continuous geometry of elements a, then (i) in L there is a vector-valued dimension function $D(a;\mathfrak{P})$ with $\mathfrak{P}$ varying over the maximal ideals of the centre of L, (ii) L is isomorphic to part of a direct sum of irreducible continuous geometries. The present paper gives a corresponding theorem for a reducible continuous regular ring $\mathfrak{R}$ of elements α: (i) in $\mathfrak{R}$ there is a vector-valued rank function $R(\alpha;\mathfrak{s})$ with $\mathfrak{s}$ varying over the maximal two-sided ideals of $\mathfrak{R}$, (ii) $\mathfrak{R}$ is isomorphic to part of a direct sum of irreducible continuous regular rings. *I. Halperin* (Kingston, Ont.)

Gel'fand, I. M., and Fomin, S. V. Unitary representations of Lie groups and geodesic flows on surfaces of constant negative curvature. Doklady Akad. Nauk SSSR (N.S.) **76**, 771–774 (1951). (Russian)

The authors consider the spectrum of a geodesic flow on a surface of constant negative curvature. They show that this spectrum in the case of a 2-dimensional surface is a Lebesgue spectrum (i.e. the spectral measures are all equivalent to the ordinary Lebesgue measure). In case the surface is compact they show that the spectrum is an enumerably multiple Lebesgue spectrum. The well known theorems of Hopf and Hedlund [cf. e.g. E. Hopf, Ber. Verh. Sächs. Akad. Wiss. Leipzig **91**, 261–304 (1939); these Rev. **1**, 243] on the metric transitivity and mixing properties of geodesic flows on surfaces of constant negative curvature follow as corollaries.

The method used to show that the spectrum is a Lebesgue spectrum is to represent the geodesic flow as a flow defined on the co-set space G/N of the group G of real matrices of order 2 with determinant 1 modulo a suitable discrete subgroup N. The flow S_t is defined by means of multiplication by $\left(\begin{smallmatrix} e^{-t} & 0 \\ 0 & e^{+t}\end{smallmatrix}\right)$. The authors then appeal to the classification of irreducible unitary representations of the group G [cf. I. M. Gelfand and M. A. Naĭmark, Izvestiya Akad. Nauk. SSSR. Ser. Mat. **11**, 411–504 (1947); these Rev. **9**, 495]. They show that for each type of these representations the spectrum is a Lebesgue spectrum. Their result follows as a consequence.

By similar methods one can compute the spectrum of a flow defined on the co-set space G/N of any locally compact Lie group G modulo a discrete subgroup N. The flow will be defined by a 1-parameter subgroup s_t of G provided the irreducible unitary representations of G are known. Modifying their method the authors deduce that the spectrum of a geodesic flow on a surface of constant negative curvature of arbitrary dimension is an absolutely continuous spectrum (i.e. the spectral measures are absolutely continuous set functions). Proofs are either omitted or only sketched. *Y. N. Dowker* (Manchester).

Barbašin, E. A. On homomorphisms of dynamical systems. II. Mat. Sbornik N.S. **29(71)**, 501–518 (1951). (Russian)

[For part I see Mat. Sbornik N.S. **27(69)**, 455–470 (1950); these Rev. **12**, 422.] This paper concerns sections of dy-

namical systems, covering dynamical systems and their connections with the author's K-homomorphisms and I-homomorphisms which are essentially homomorphisms of a dynamical system onto the circle group and line group. Typical result: If a dynamical system in a simply connected space has a K-homomorphism, then it has a I-homomorphism such that the I-section is a component of the K-section.

W. H. Gottschalk (Philadelphia, Pa.).

Calculus of Variations

✶Morse, Marston. Recent advances in variational theory in the large. Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 143–156. Amer. Math. Soc., Providence, R. I., 1952.

The author sketches the modifications (towards greater abstractness and generality) which his theory in the large has undergone from 1936, date of the Oslo Congress, to 1940 and subsequently in later papers [Ann. of Math. (2) **41**, 419–454 (1940); these Rev. **1**, 320; Fund. Math. **35**, 47–78 (1948); these Rev. **10**, 391]. He announces further a generalized form of a theorem of Lyusternik and Schnirelmann on the dimension of a critical set. Advances due to other mathematicians are mentioned briefly.

L. C. Young (Madison, Wis.).

✶Janet, Maurice. Sur le "calcul des variations." Congrès International de Philosophie des Sciences, Paris, 1949, vol. III, Philosophie Mathématique, Mécanique, pp. 19–22. Actualités Sci. Ind., no. 1137. Hermann & Cie., Paris, 1951.

Lyusternik, L. A., and Fet, A. I. Variational problems on closed manifolds. Doklady Akad. Nauk SSSR (N.S.) **81**, 17–18 (1951). (Russian)

Let R be a closed n-dimensional manifold (which is supposed four times differentiable) and let J denote a regular positive variational problem concerning piecewise continuously differentiable curves on R. The authors prove: (1) there exist at least two distinct extremals of J which join two arbitrarily given points of R; (2) there exists a closed extremal of J on R. No reference is made to work on these topics outside Russia.

L. C. Young (Madison, Wis.).

Kerimov, M. K. On necessary conditions for an extremum in discontinuous variational problems with variable end-points. Doklady Akad. Nauk SSSR (N.S.) **79**, 565–568 (1951). (Russian)

Kerimov, M. K. On Jacobi's condition in discontinuous variational problems with variable end-points. Doklady Akad. Nauk SSSR (N.S.) **79**, 719–722 (1951). (Russian)

Let L_1, L_0, L_2 be three given simple unbounded continuous curves of the (x, y)-plane dividing the plane into four unbounded parts D_0, D_1^-, D_1^+, D_2. Let $\{C\}$ be the family of all continuous curves, sums of two arcs C^-, C^+, $C^-\subset D_1^-$, $C^+\subset D_1^+$, where C^- joins a point 1 on L_1 with a point 0 on L_0, and C^+ joins the point 0 with a point 2 on L_2. Let $F^-(x, y, y')$, $F^+(x, y, y')$ be any two functions defined respectively for all $(x, y)\epsilon D_1^-$, $-\infty<y'<+\infty$, and $(x, y)\epsilon D_1^+$, $-\infty<y'<+\infty$. The curves L_1, L_0, L_2 are supposed to be represented parametrically by functions continuous together with their derivatives of third order, the functions F^-, F^+ are supposed to be continuous with their partial derivatives of fourth order. The methods and notations of G. A. Bliss [Calculus of Variations, Open Court, Chicago, 1925] are used [a Russian translation of 1950 of this book is quoted] as well as the quadratic forms of M. Morse's theory [The Calculus of Variations in the Large, Amer. Math. Soc. Colloq. Publ., v. 18, New York, 1934]. The curves C have an angular point at 0 and satisfy the usual conditions as stated by Bliss [loc. cit.]. Let J be the functional $J=\int_{C^-}F^-(x, y, y')dx+\int_{C^+}F^+(x, y, y')dx$. The author studies the necessary conditions for an extremum of the functional J in the family $\{C\}$ where the points 1, 0, 2 are variable points of the curves L_1, L_0, L_2. Besides the Euler equation, the two transversality conditions (end conditions), the corner condition at the point 0, the author gives the Jacobi condition in terms of the second variation (first paper) as well as in terms of the focal point (second paper) and in terms of eigenvalues. Sufficient conditions are given for a strong relative minimum.

L. Cesari (Lafayette, Ind.).

Giltay, J. On a variational problem. Scienca Revuo **3**, 48–49 (1951). (Esperanto)

The author considers the maximum of $|I|$, where $I=\int y dx\cdot\int yx dx/\int y^2 dx$, the integrations are over $[-1, 1]$, and $y=y(x)$ is (i) real-valued, or (ii) non-negative. He presumes that the maxima are attained and shows that the extremals for case (i) must be proportional to $1\pm x\sqrt{3}$, which gives $|I|=1/\sqrt{3}$, and those for (ii) to $|1\pm 2x|+1\pm 2x$, which gives $|I|=9/16$. He notes that case (i) can also be settled by using Legendre polynomials. [A rigorous treatment of (ii) is obtainable by substituting $x=\pm(X-1)$ and applying results given recently by G. Aumann, Acta Univ. Szeged. Sect. Sci. Math. **13**, 163–168 (1950), especially §§1.4 and 3.2; these Rev. **12**, 838.]

H. P. Mulholland.

Sigalov, A. G. On the oscillation of a stationary function of a quadratic double integral. Doklady Akad. Nauk SSSR (N.S.) **81**, 505–508 (1951). (Russian)

The author establishes the following theorem: Let the function $Z(x, y)\epsilon A^2$ in a closed region $\bar{D}$ of the (x, y)-plane (that is, Z is to be absolutely continuous in Tonelli's sense in $\bar{D}$ and Z_x^2 and Z_y^2 are to be summable in $\bar{D}$), and let Z satisfy the variational condition

$$\iint_G \{(aZ_x+bZ_y+dZ+r)\eta_x+(bZ_x+cZ_y+eZ+s)\eta_y+(dZ_x+eZ_y+fZ+t)\eta\}dxdy=0$$

whenever the function $\eta\epsilon A^2$ in $\bar{D}$ and $\eta=0$ in $\bar{D}-G$, for any region $\bar{G}\subset D$. Suppose, further, that for any region $\bar{D}_1\subset D$,

(i) $$\iint_{D_1}\{|d|^{\gamma_1}+|e|^{\gamma_1}+|f|^{\gamma_2}+|r|^{\gamma_3}+|s|^{\gamma_3}+|t|^{\gamma_4}\}dxdy\leqq L_1,$$

where $\gamma_1>2$, $\gamma_2>1$, $\gamma_3>2$, $\gamma_4>1$,

(ii) $$\sup_{(x,y)\epsilon D_1'}|Z(x, y)|\leqq L_2;$$

(iii) $$aZ_x^2+2bZ_xZ_y+cZ_y^2\geqq m(Z_x^2+Z_y^2),$$

where $m>0$, for any values Z_x, Z_y and for almost all $(x, y)\epsilon D_1$, (iv) $\iint_{D_1}(Z_x^2+Z_y^2)^{\frac{1}{2}}dxdy\leqq L_1$. Then $|Z(x, y)|\leqq L_3$, $(x, y)\epsilon D_1$, where L_3 is a constant depending only on m, γ_1, γ_2, γ_3, γ_4, L_1, L_2 and not on the coefficients a, b, $\cdots$ and the region D_1.

H. P. Mulholland (Birmingham).

Theory of Probability

Ghosh, Birendranath. Random distances within a rectangle and between two rectangles. Bull. Calcutta Math. Soc. **43**, 17–24 (1951).

Computation of the distribution and lower moments of the distance between two points independently drawn with the natural probabilities from two (not necessarily distinct or disjoint) parallel rectangles in the plane is asserted to have some practical importance. A general attack is illustrated. *L. J. Savage* (Paris).

Krishna Iyer, P. V., and Sukhatme, B. V. Probability distribution of points on a line. Science and Culture **15**, 200 (1949).

Tortrat, Albert. Divisibilité des "lois de probabilité convexes." C. R. Acad. Sci. Paris **233**, 914–915 (1951).

This paper states several conditions which are sufficient, and another condition which is necessary, for the decomposability of what the author calls convex probability distribution functions. A distribution is decomposable if it is the convolution of two distribution functions (the trivial case being excluded). A "convex" probability distribution is one whose characteristic function $\varphi(t)$ is real, even, continuous, convex for $t>0$, and approaches zero as $t\to\infty$. Polya [Proc. First Berkeley Symposium, pp. 115–123, University of California Press, 1949; these Rev. **10**, 463] proved that such distributions have density functions symmetric about the origin. *J. Wolfowitz* (Ithaca, N. Y.).

Ginzburg, G. M. On uniqueness conditions for limit distributions. Izvestiya Akad. Nauk SSSR. Ser. Mat. **15**, 563–580 (1951). (Russian)

"This paper establishes necessary and sufficient conditions for the uniqueness of the limiting distribution (as $t\to\infty$) determined by the stochastic equation

$$\Delta y = A(y)\Delta t + f(\alpha, y)\sqrt{\Delta t},$$
$$Ef^2(\alpha, y) = B(y),$$

where $A(y)$ and $B(y)$, are analytic functions of y for all real y". *From the author's summary.*

Chung, K. L., and Wolfowitz, J. On a limit theorem in renewal theory. Ann. of Math. (2) **55**, 1–6 (1952).

Let $\{X_n, n\geqq 1\}$ be a sequence of mutually independent integer-valued random variables with a common distribution, and suppose that $0<E\{X_1\}\leqq\infty$. Let t be the (positive) greatest common divisor of the indices n such that $P\{X_1=n\}>0$. Then it is proved that, if $S_j=\sum\{X_i,$

$$\sum_1^\infty P\{S_j=n\}\to\begin{cases} t/E\{X_1\} & \text{if } n\to\infty,\\ 0 & \text{if } n\to-\infty.\end{cases}$$

This theorem was proved by Erdös, Feller and Pollard [Bull. Amer. Math. Soc. **55**, 201–204 (1949); these Rev. **10**, 367] for the case $X_1\geqq 0$. In this case it can also be referred back to theorems on Markov chains due to Kolmogorov [Mat. Sbornik N.S. 1(43), 607–610 (1936)]. *J. L. Doob.*

✓★**Doob, J. L. Continuous parameter martingales.** Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950, pp. 269–277. University of California Press, Berkeley and Los Angeles, 1951. **$11.00.**

Let T be a simply ordered set. A family of random variables $\{x(t), t\epsilon T\}$ indexed by T is a martingale if $E\{|x(t)|\}<\infty$ for $t\epsilon T$ and $E\{x(t) \mid x(r), r\leqq s\}=x(s)$ with probability 1, if $s<t$. For a martingale $\{x_j, j\leqq n\}$ with a finite number of random variables the following result is proved. Let $r_2>r_1$ be real numbers and β be the random variable defined as the number of times the graph determined by the sample sequence $x_1, \cdots, x_n$ passes from below r_1 to above r_2. Then $E\{\beta\}\leqq E\{|x_n-r_1|\}/(r_2-r_1)$. The author uses this result to prove and to show the connection between the following two theorems. First, if $\{x(n), n\geqq 1\}$ is an integral parameter martingale and if $\sup_n E\{|x(n)|\}<\infty$, then the random variables of the process converge with probability one to a finite limit. Secondly, if $\{x(t), a\leqq t\leqq b\}$ is a continuous parameter martingale and stochastic measures are defined suitably, then almost all sample functions of the process have finite left and right hand limits everywhere on $[a, b]$. This improves a result obtained by the author in a previous paper [Trans. Amer. Math. Soc. **47**, 455–486 (1940); these Rev. **1**, 343]. The author indicates how to apply this martingale result to obtain continuity properties of sample functions of processes with independent increments and Markov processes. *J. L. Snell* (Princeton, N. J.).

✓★**Cramér, Harald. A contribution to the theory of stochastic processes.** Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950, pp. 329–339. University of California Press, Berkeley and Los Angeles, 1951. **$11.00.**

Let $\{z(t), -\infty<t<\infty\}$ be a family of random variables, with $E\{z(t)\}=0$, $E\{z(t)\overline{z(u)}\}=\rho(t, u)$, where ρ is of bounded variation in bounded domains, so that $\int\int_A d^2\rho(t, u)$ is a complex-valued measure of Borel sets A. The integral $\xi(g)=\int g(t)dz(t)$ is defined in the obvious way, and is shown to have the formally obvious properties. In particular, $Z(S)=\int_S dz(t)$ is an additive random set function, and the author writes (1) $\xi(g)=\int g(t)dZ$. Conversely, if $Z(S)$ is an additive random set function with $E\{Z(S)\}=0$, a corresponding $z(t)$ family and covariance function ρ can be determined. Suppose that ρ is of bounded variation in the above sense. Then (1) maps a certain class $\Lambda(\rho)$ of functions g isometrically into the (mean square) closed linear manifold of random variables generated by the $Z(S)$'s. Finally, suppose that $g(t, \lambda)\epsilon\Lambda(\rho)$ for each t, and define (2) $x(t)=\int g(t, \lambda)dZ$. Then (3) $E\{x(t)\overline{x(u)}\}=\int\int g(t, \lambda)g(u, \mu)d^2\rho(\lambda, \mu)$. Conversely, if the covariance function of an $x(t)$ family has the form (3), where ρ is of bounded variation in every finite domain, there exists a $Z(S)$ with the above described properties such that $x(t)$ is given by (2). This result generalizes one due to Karhunen in which ρ is the covariance function of a process with orthogonal increments [Ann. Acad. Sci. Fennicae Ser. A. I. Math. Phys. no. **37** (1947); these Rev. **9**, 292]. Loève considered the case in which ρ is of bounded variation over the whole plane and $g(t, \lambda)=e^{it\lambda}$ [Supplement to Lévy, Processus stochastiques et mouvement brownien, Gauthier-Villars, Paris, 1948; these Rev. **10**, 551]. *J. L. Doob.*

Kallianpur, Gopinath. Intégrale de Stieltjes stochastique et un théorème sur les fonctions aléatoires d'ensembles. C. R. Acad. Sci. Paris **232**, 922–923 (1951).

Let $x(t)$ be a real-valued random function of the real variable t, such that $Ex(t)=0$, while the covariance function $r(s, t)=Ex(s)x(t)$ is of bounded variation over any finite domain in the (s, t)-plane. It is then known that there exists an additive random set function $Z(S)$, defined for any bounded Borel set S on the real axis, and such that $Z(S)=x(b)-x(a)$ whenever S is an interval (a, b). For the

proof of this property the author refers to the paper reviewed just above, where, however, it is not claimed as the author states that $Z(S)$ plays the part of spectral function of $x(t)$. The author gives a definition of the integral $\int f(a)dZ$, which is analogous to the definition of the integral $\int f(a)x(t)dt$ given by Karhunen [Ann. Acad. Sci. Fennicae. Ser. A. I. Math. Phys. no. **37** (1947); these Rev. **9**, 292]. He then states without proof a theorem concerning the representation of $Z(S)$ in the form of a certain Karhunen integral. The conditions of this theorem given in the note will, however, require some modification, since the theorem as given by the author does not hold, e.g., in the case when $Z(S)$ is the random set function attached to an orthogonal and homogeneous $x(t)$. *H. Cramér* (Stockholm).

★**Kakutani, Shizuo. Random ergodic theorems and Markoff processes with a stable distribution.** Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950, pp. 247–261. University of California Press, Berkeley and Los Angeles, 1951. $11.00.

Let $(S\times X, \mathfrak{B}\otimes\mathfrak{C}, m\times\mu)$ be the direct product probability space of two probability spaces $(S, \mathfrak{B}, m)$ and $(X, \mathfrak{C}, \mu)$. Let there be associated to each $x\epsilon X$ a $\mathfrak{B}$-m-measure preserving mapping $\varphi_x(s)$ of S onto S. The family $\Phi=\{\varphi_x|x\epsilon X\}$ is assumed to be $(\mathfrak{B},\mathfrak{C})$-measurable in the sense that $B\epsilon\mathfrak{B}$ implies $\{(s, x)|\varphi_x(s)\epsilon B\}\epsilon\mathfrak{B}\otimes\mathfrak{C}$. Let further $(\Omega, \mathfrak{C}^*, \mu^*)$ be the infinite direct product probability space:

$$\Omega=\mathop{\dot{\mathsf{P}}}_{n=-\infty}^{\infty} X_n\ (X_n=X),\quad \mathfrak{C}^*=\mathop{\dot{\mathsf{P}}}_{n=-\infty}^{\infty}\otimes\mathfrak{C}_n\ (\mathfrak{C}_n=\mathfrak{C}),$$

$$\mu^*(\mathop{\dot{\mathsf{P}}}_{n=-\infty}^{\infty} E_n)=\prod_{n=-\infty}^{\infty}\mu(E_n),\quad E_n\epsilon\mathfrak{C}_n.$$

Denoting by $x_n(\omega)$ the nth coordinate of the point $\omega\epsilon\Omega$, the random ergodic theorem of Ulam and von Neumann [abstract, Bull. Amer. Math. Soc. **51**, 660 (1945)] may be generalised as follows: For any $f(s)\epsilon L_p(S)$ $(p\geqq 1)$, there exists an $\mathfrak{C}^*$-μ^*-null set N^* of Ω such that, for any $\omega\epsilon\Omega-N^*$, there exists a function $\bar{f}_\omega(s)\epsilon L_p(S)$ such that

$$\lim_{n\to\infty}\|n^{-1}\sum_{k=0}^{n-1}f(\varphi_{x_{k-1}(\omega)}\cdots\varphi_{x_0(\omega)}(s))-\bar{f}_\omega(s)\|_{L_p(S)}=0$$

and

$$\lim_{n\to\infty}n^{-1}\sum_{k=0}^{n-1}f(\varphi_{x_{k-1}(\omega)}\cdots\varphi_{x_0(\omega)}(s))=\bar{f}_\omega(s)$$

$\mathfrak{B}-m$-almost everywhere on S. The proof is reduced to the usual mean and individual ergodic theorems for the $(\mathfrak{B}\otimes\mathfrak{C}^*)$-$m\times\mu^*$-measure preserving mapping

$$\varphi^*(s,\omega)=[\varphi_{x_0(\omega)}(s), \psi(\omega)]$$

of the direct product probability space $(S\times\Omega, \mathfrak{B}\otimes\mathfrak{C}^*, m\times\mu^*)$ on itself, where $\psi(\omega)$ denotes the shift transformation of Ω: $x_n(\psi(\omega))=x_{n+1}(\omega)$. The family Φ defines, by

$$P(s, B)=\mu\{x|\varphi_x(s)\epsilon B\},$$

the transition probability of a simple Markoff process such that $s\epsilon S$ is transferred into $B\epsilon\mathfrak{B}$ after the elapse of unit time. This induced Markoff process $P(s, B)$ has the stable distribution $m(B)$: $\int_S P(s, B)m(ds)=m(B)$. The known results [the reviewer, Proc. Imp. Acad. Tokyo **16**, 43–48 (1940); the author, ibid. **16**, 49–54 (1940); these Rev. **1**, 343] concerning the Markoff process with a stable distribution, are, for this particular $P(s, B)$, thus proved to be nothing but the "integrated form" of the random ergodic theorem for the family Φ [cf. H. Anzai, Osaka Math. J. **2**, 43–49 (1950); these Rev. **12**, 190]. By making use of this fact, the equivalence of various conditions of ergodicity for the family Φ is discussed. *K. Yosida* (Nagoya).

★**Lévy, Paul. Wiener's random function, and other Laplacian random functions.** Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950, pp. 171–187. University of California Press, Berkeley and Los Angeles, 1951. $11.00.

1) Un processus de Wiener-Levy $X(t)$ $(0\leqq t\leqq 2\pi)$ est representable par la formule de Wiener:

$$X(t)=\frac{\xi' t}{(2\pi)^{\frac{1}{2}}}+\sum_1^\infty\frac{1}{n\pi^{\frac{1}{2}}}[\xi_n(\cos nt-1)+\xi_n'\sin nt]$$

processus stochastique, où ξ', ξ_n', ξ_n sont des variables aléatoires laplaciennes mutuellement indépendantes; l'auteur utilise cette formule pour déterminer la caractéristique de $\int_0^1 X^2(u)du$ et celle de l'aire comprise entre l'arc et la corde d'une trajectoire de mouvement brownien plan. 2) Sous des conditions tres générales, un processus laplacien complexe $X(t)$ débutant à l'instant t_0 $(t_0\geqq-\infty)$ satisfait à:

$$dX(t)\sim dt\int_{t_0}^t F(t, u)dX(u)+\xi\sigma(t)\sqrt{dt},$$

où $F(t, u)$ est une certaine fonction et ξ une variable aléatoire laplacienne indépendante de $X(u)$ pour $u\leqq t$; l'auteur détermine la convariance $\Gamma(t, u)$ de $X(t)$ connaissant $F(t, u)$ et $\sigma(t)$, et réciproquement $F(t, u)$ connaissant $\Gamma(t, u)$ (ce deuxième problème n'est résolu que sous certaines conditions; $F(t, u)$ est alors déterminé par une équation intégrale). [Remarque du rapporteur: ces résultats de l'auteur se déduisent aussi du fait qu'en général un processus laplacien est limite en loi d'une fonction aléatoire dérivée d'un processus de Poisson.] 3) Un processus Laplacien $X(t)$ $(0\leqq t\leqq 2\pi)$ admet un développement en série de Fourier (en général divergent): $X(t)\sim\sum A_n e^{int}$; l'auteur montre que les variables aléatoires laplaciennes A_n sont mutuellement indépendantes si et seulement si $X(t)$ est stationnaire, et envisage quelques cas particuliers. *R. Fortet* (Caen).

★**Feller, William. Some recent trends in the mathematical theory of diffusion.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 322–339. Amer. Math. Soc., Providence, R. I., 1952.

This address "outlines some new results and open problems concerning diffusion theory where we find an intimate interplay between differential equations and measure theory in function space". Let $u(y; t, x)$ denote the transition probability that x is transferred, by a temporally homogeneous Markoff process on the real line, to y after the elapse of t units of time. Under a certain "continuity condition of Lindeberg's type", $u(y; t, x)$ satisfies the backward diffusion equation (i) $z_t(t, y)=b(y)z_{yy}(t, y)+a(y)z_y(t, y)$ (A. Kolmogoroff and W. Feller). Thus it is usually expected that $u(y; t, x)$ should satisfy the forward (Fokker-Planck's) diffusion equation

(ii) $$w_t(t, x)=\{b(x)w(t, x)\}_{xx}-\{a(x)w(t, x)\}_x.$$

It is stressed that there exist cases where $u(y; t, x)$ satisfies only (i); instead of (ii) a positive operator is to be added to the right side. This remarkable fact was recognized after Feller's investigation of a singular diffusion equation connected with Sewall Wright's theory of evolution and Doob's

discovery and explanation, for the case of a discrete Markoff process, of the fact that there are cases when $\sum_y u(y;t,x)<1$. In this way, the existence and uniqueness of the solutions of (i) and (ii) by semi-group theory (K. Yosida and E. Hille) or by the theory of stochastic differential equations (K. Itô) is shown to be associated with special cases and with special boundary conditions. For example, the classical Brownian motion on $(-\infty, \infty)$ (N. Wiener and P. Lévy) corresponds to the phenomenon of natural boundaries, where no boundary conditions at $\pm\infty$ are imposed, since almost all the paths of the particle are continuous and hence the particle cannot reach infinity. Thus, besides the classical boundary conditions (absorbing and elastic) a new kind of boundary condition is introduced: a particle, when it reaches the boundary, stays for a while and is then transferred back into the interior with a certain randomness. In more than one dimension, Ornstein-Uhlenbeck's refined model of diffusion in which the particle has finite velocity (but no acceleration) is interesting. As was shown by Doob, the position of the particle is not Markovian, but the conjugate pair (position, velocity) becomes Markovian. In this connection, the calculations of various Wiener's functionals due to Cameron, Martin and Kac are explained. Also, Feynman's approach to quantum physics may open a new chapter of Markoff processes. Finally, Bochner's research on the stationary processes as solutions of $z_t=Az$, where the operators A denote certain fractional powers of d^2/dx^2, is investigated from the view point of M. Riesz' potential

$$I^\alpha f(x)=[2\Gamma(\alpha)\cdot\cos\pi\alpha/2]^{-1}\int_{-\infty}^{\infty}f(y)|y-x|^{\alpha-1}dy.$$

For, by $I^{-2}=-d^2/dx^2$, the generalized diffusion equation $z_t=-I^{-\alpha}z$ is naturally introduced. *K. Yosida.*

***Kampé de Fériet, J. Sur l'analyse spectrale d'une fonction stationnaire en moyenne.** Actes du Colloque International de Mécanique, Poitiers, 1950. Tome III. Étude sur la mécanique des fluides. pp. 317–335. Publ. Sci. Tech. Ministère de l'Air, Paris, no. 251 (1951).
Expository lecture. *J. L. Doob* (Urbana, Ill.).

✓***Wiener, Norbert. Comprehensive view of prediction theory.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 308–321. Amer. Math. Soc., Providence, R. I., 1952.
Expository lecture. *J. L. Doob* (Urbana, Ill.).

Callen, Herbert B., and Welton, Theodore A. Irreversibility and generalized noise. Physical Rev. (2) 83, 34–40 (1951).

The Nyquist relation for resistance noise is generalized to give the mean squared value of a fluctuating generalized force V in a quantum mechanical system with closely spaced energy levels. The expectation of V^2 is expressed in the form $(2/\pi)\int R(\omega)E(\omega,T)d\omega$ where $R(\omega)$ is the real part of a suitably defined impedance function, T is the absolute temperature, and

$$E(\omega,T)=\tfrac{1}{2}\hbar\omega+\hbar\omega[\exp(\hbar\omega/kT)-1]^{-1}.$$

Applications are made to Brownian motion, electric field fluctuations in a vacuum, and pressure fluctuations in a gas. *E. N. Gilbert* (Murray Hill, N. J.).

Vallée, Robert. Interprétation géométrique, dans l'espace de Hilbert, de propriétés de régimes périodiques ou impulsionnels. Ann. Télécommun. 6, 61–66 (1951).
Expository paper.

Loeb, Julien. Exposé simplifié de la théorie "informationnelle" de Shannon en vue de son application aux problèmes de télécommande, télémesure ou mesure. Ann. Télécommun. 6, 67–76 (1951).
Expository paper.

Loeb, Julien. Introduction aux théories du signal et de l'information. Ann. Télécommun. 5, 246–254 (1950).
Expository paper.

✓***Dubourdieu, J. Théorie mathématique des assurances. Fasc. I. Théorie mathématique du risque dans les assurances de répartition.** Gauthier-Villars, Paris, 1952. xx+306 pp. 3500 francs.

This is the first volume of a new treatise on actuarial mathematics and is devoted to the theory of risk. This fact already indicates that the author is seeking a new approach to the subject. It is also clear that he is aiming at a reform of the traditional training of actuarial students.

The first chapter gives an outline of the scope of the book. The nature of the insurance contract is discussed and it is stated that this contract deals with random events. It is therefore appropriate to approach the subject of actuarial mathematics from a probabilistic view point. This is in contrast with the conventional treatment of actuarial mathematics which overemphasizes the purely financial aspects of the insurance contract and deals mainly with financial operations. The author excludes from his consideration all operations of capitalization and concentrates on those types of insurance in which the random element predominates. Therefore he hardly considers life insurance but studies mainly forms of insurance other than life insurance such as, for example, automobile insurance or fire insurance. In the second and third chapter the author discusses the application of probability theory to the theory of insurance and presents the classical theory of risk. The risk insured is considered as a random variable, the concepts of net premium and of security charges and of security reserves are introduced. The maximum loss a company can take is discussed and its relation to security charges and office premiums is studied. The presentation of the classical theory of risk differs considerably in form from the treatment of the subject in books on actuarial mathematics. Yet the problems are approached from the viewpoint of an individual insurance policy and with the attitude which we find in the discussions of the classical theory of risk in life insurance. It is therefore completely justified to call this the classical theory of risk even though it does not contain the familiar theorems and discussions which one usually associates with the term. Non-life insurance is classified into two main groups; the first with a well defined insurance value and the second without it. The hypothesis of constant risk leads to a first discussion of Poisson's law. (It is treated in greater detail in an appended note.) Chapter four introduces the principles of reinsurance in close connection with the theory of risk. The security coefficient is defined as well as different modes of reinsurance. The effects of quota reinsurance as well as of reinsurance of the sum in excess of own retention is studied.

In chapter five the author first considers a gambler who plays an infinite sequence of inequitable games. It is assumed that the odds favor the gambler and that he has a certain initial fortune K. Using an idea of B. de Finetti [Giorn. Ist. Ital. Attuari 10, 41–51 (1939)] the author establishes a close connection between the theory of risk and the problem of the gambler's ruin. The theorem of de Finetti

states that e^{-rK} is an upper bound for the probability of the gambler's ruin; here r is a constant, characteristic for the game, which is called the security index. This quantity r is obtained as the positive solution of the equation $\int_{-\infty}^{+\infty} e^{-rz}dF(x)=1$, where $F(x)$ is the distribution function of the random variable representing the gain (or loss) in one round of the game. Under certain, not too restrictive, assumptions this equation has only one positive root. This theorem is then applied to the theory of risk by considering an insurance contract as a game favoring the insurance company. The premium charges are then determined so that the security index should assume a fixed value for all contracts. The probability of ruin is then at most equal to e^{-rK}. This approach is applied to the study of the risk fund and of the reserves of the insurance company. This method essentially determines a plan for the financing of the security reserve in such a manner that the probability of ruin can never exceed a prearranged value. The author gives then a detailed discussion of reinsurance from this viewpoint. Chapter six presents a good account of the collective theory of risk. This theory is not concerned with the individual insurance but considers the development of the risk fund as a discontinuous stochastic process. The basic concepts of this theory are introduced and an upper bound for the risk function is determined. Finally the author compares the collective theory of risk with the theory discussed in chapter five and shows that these theories are related.

In the reviewer's opinion the present volume is a valuable contribution to actuarial literature. It is not only the first text book which covers the collective theory of risk but it opens also a new field to the actuary by demonstrating that actuarial problems can be found also outside of life insurance and can be attacked by mathematical methods.

E. Lukacs (Washington, D. C.).

Mathematical Statistics

Recent advances in mathematical statistics. Bibliography, 1943–47. J. Roy. Statist. Soc. Ser. A. 114, 497–558 (1951).

"This bibliography is a continuation of that compiled by Miss Rigg for 1940–42 [same J. 109, 395–450 (1946)] and covers papers, published during the years 1943–47, which deal with the development of some aspect of statistical theory, or the application of statistical method to practical problems." *Extract from the paper.*

Azorín, F., and Wold, H. Product sums and modulus sums of H. Wold's normal deviates. Trabajos Estadística 1, 5–28 (1950). (English and Spanish)

Recently Wold [Random normal deviates, Cambridge, 1948; these Rev. 10, 553] published a table containing 25,000 random normal deviates. These occupy 50 pages, with 10 columns of 50 numbers each on a page. If x_{ik}, x_{jk} are the numbers on line k of columns i and j, respectively, on the same page, the present table gives the values of

$$W_{ij}=\sum_{k=1}^{50} x_{ik}x_{jk};\quad V_i=\sum_{k=1}^{50}|x_{ik}|.$$

Tests as to whether the W_{ij} and the V_i follow their expected theoretical distributions are presented. The product sums are intended to facilitate artificial sampling experiments relating to the distribution of correlation coefficients. The modulus sums, which were a by-product, appear to have less immediate utility. *W. G. Cochran.*

Nabeya, Seiji. Note on the moments of the transformed correlation. Ann. Inst. Statist. Math., Tokyo 3, 1 (1951).

Cansado, Enrique. A systematic exposition of Pearson's distribution. Trabajos Estadística 1, 279–296 (1950). (Spanish and English)

Another classification of the type I, III, IV, V, VI and normal frequency functions of the Pearson system according to the forms to which one reduces their integrals.

C. C. Craig (Ann Arbor, Mich.).

Cansado, Enrique. Logarithmico-Pearson distributions. Trabajos Estadística 1, 297–313 (1950). (Spanish and English)

The author uses the fact that the characteristic function of the logarithm of a random variable is at once expressible as the expected value of a power of the variable to write such characteristic functions when the variable obeys one of the Pearson frequency functions discussed in the paper reviewed above. He applies these in the obvious way to find familiar distributions of products and quotients of such variables. *C. C. Craig* (Ann Arbor, Mich.).

Nanda, D. N. Probability distribution tables of the larger root of a determinantal equation with two roots. J. Indian Soc. Agric. Statistics 3, 175–177 (1951).

Let $x=\|x_{ij}\|$, $x^*=\|x^*_{ij}\|$ be two p-variate sample matrices with V_1 and V_2 degrees of freedom, $S=\|xx'\|/V_1$, $S^*=\|x^*x^{*\prime}\|/V_2$, the covariance matrices which under the null hypothesis are independent estimates of the same population covariance matrix; then the distributions of the largest, smallest, and any intermediate roots of the determinantal equation $|A-\theta(A+B)|=0$, where $A=V_1S$, $B=V_2S^*$, have been obtained by the author for $l=2, 3, 4$, and 5, $l=\min(p, V_1)$, $m=\frac{1}{2}(|p-V_1|+1)$, $n=\frac{1}{2}(V_2-p-1)$ [Ann. Math. Statistics 19, 47–57 (1948); these Rev. 9, 453]. In the present paper for the particular case $l=2$, the author tabulates the value of θ_1 to two significant figures which are used in tests of significance of the larger root at the 5% and 1% levels of significance with $m=0(.5)2$ and $n=.5(.5)10$.

L. A. Aroian (Culver City, Calif.).

Krishnamoorthy, A. S. Multivariate binomial and Poisson distributions. Sankhyā 11, 117–124 (1951).

The multivariate binomial distribution is expanded in terms of G-polynomials used already by Aitken and Gonin in the bivariate case [Proc. Roy. Soc. Edinburgh 55, 114–125 (1935)]. A similar expansion in terms of Charlier polynomials is obtained for the multivariate Poisson distribution as the limit of the multivariate binomial distribution. The expansions are obtained by applying transformations to the respective factorial moment generating functions.

G. E. Noether (Boston, Mass.).

Krishnamoorthy, A. S., and Parthasarathy, M. A multivariate gamma-type distribution. Ann. Math. Statistics 22, 549–557 (1951).

It was shown by Kibble [Sankhyā 5, 137–150 (1941); these Rev. 4, 103] that a bivariate frequency function in which each variable has identical gamma type marginal distributions is expressible in a series in products of Laguerre polynomials. The present authors extend this result to the

multivariate case both when the variables all have identical gamma type marginal distributions and more generally when the variables have gamma type marginal distributions with different parameters. They establish sufficient conditions for the convergence of the series so found.

C. C. Craig (Ann Arbor, Mich.).

Leslie, P. H. The calculation of χ^2 for an $r\times c$ contingency table. Biometrics 7, 283–286 (1951).

Zinger, A. A. On independent samples from normal populations. Uspehi Matem. Nauk (N.S.) 6, no. 5(45), 172–175 (1951). (Russian)

Let $X_1, \cdots, X_n$ be n independent, identically distributed chance variables, $\bar{X}=n^{-1}\sum X_i$, $S^2=n^{-1}\sum(X_i-\bar{X})^2$. If $\bar{X}$ and S^2 are independently distributed then X_i is normally distributed. This result has been proved earlier by Kawata and Sakamoto [J. Math. Soc. Japan 1, 111–115 (1949); these Rev. 11, 188] and, under the assumption of a finite second moment, by Lukacs [Ann. Math. Statistics 13, 91–93 (1942); these Rev. 4, 16].

J. Wolfowitz.

McMillan, Brockway. Spread of minima of large samples. Ann. Math. Statistics 20, 444–447 (1949).

Let w be the spread of the K minima of K samples (each of size N) of values of x, where x has a (continuous) cumulative distribution function (c.d.f.) $F(x)$. Let $P_N(w)$ be the c.d.f. of w. The two main theorems proved by the author are: (i) "If $\liminf_{x\to-\infty} F(x)/[F(x+s)]=l$, then $\limsup_{N\to\infty} P_N(s)\leqq(1-l)^{K-1}$." (ii) "Let $s>0$. If $F(x)=0$ for no finite x and $\limsup_{x\to-\infty} F(x)/[F(x+s)]=L$, then $\liminf_{N\to\infty} P_N(s)\geqq[e^{-\alpha L}-e^{-\alpha}]^K$ for any $\alpha>0$."

D. F. Votaw, Jr. (New Haven, Conn.).

Wald, Abraham. On the principles of statistical inference. Trabajos Estadística 2, 113–148 (1951). (Spanish)

Translation of Notre Dame Mathematical Lectures, no. 1, University of Notre Dame, 1942; these Rev. 4, 25.

✓***Hodges, J. L., Jr., and Lehmann, E. L. Some applications of the Cramér-Rao inequality.** Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950, pp. 13–22. University of California Press, Berkeley and Los Angeles, 1951. \$11.00.

The Cramér-Rao inequality for the variance of an estimator of a parameter θ (with bias) is used to obtain a criterion which implies the admissibility and/or minimaxity of point estimators for fixed sample-size n and loss function $(\theta-d)^2/q(\theta)$. The method, which involves the solution of a differential inequality, is applied to certain problems involving the binomial, Poisson, normal, and chi-square distributions, and the unique (admissible) minimax estimator is found in each case. These procedures are minimax among all sequential procedures for which $E_\theta(N)\leqq n$, and are also admissible among procedures for which, in addition, N is bounded (N is the number of observations required).

J. C. Kiefer (Ithaca, N. Y.).

Albert, G. E., and Johnson, Ralph B. On the estimation of central intervals which contain assigned proportions of a normal univariate population. Ann. Math. Statistics 22, 596–599 (1951).

Let $\bar{y}$ and s be the sample mean and standard deviation of a sample of size N from a normal population with unknown mean and variance. Let p be given, $0<p<1$, and suppose t_p is defined by $\Pr(|t|\geqq t_p)=p$, where t is a random variable having Student's distribution on $N-1$ degrees of freedom. Let $\lambda=t_p[(N+1)/N]^{\frac{1}{2}}$ and let $A(\bar{y}, s; \lambda)$ be the proportion of the underlying population included between the limits $\bar{y}-\lambda s$ and $\bar{y}+\lambda s$. It was shown by Wilks [same Ann. 12, 91–96 (1941); these Rev. 3, 9] that $E(A)=1-p$. Adapting a method of Wald and Wolfowitz [ibid. 17, 208–215 (1946); these Rev. 8, 478], the authors find an expression for the probability distribution of A which is, however, not suitable for direct computation. They therefore give a numerical procedure which is suitable for computing the minimum value of N, given numbers α, d_1, and d_2, such that $\Pr(1-p-d_1\leqq A\leqq 1-p+d_2)\geqq\alpha$. A short table is given. The treatment can be generalized to include the case where the sample variables y_i are related by linear regression to sure variables $x_{i1}, \cdots, x_{ik}$. It is stated that organized computation does not seem feasible for this case.

T. E. Harris (Santa Monica, Calif.).

Dresselaers, Céline, et Gillis, Paul P. Un test séquentiel unilatéral. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 37, 713–727 (1951).

Let $X_1, X_2, \cdots$ be independent random variables with probability density $f(x;\theta)$ where θ is real-valued. For testing the hypothesis H: $\theta=\theta_0$ against the alternatives $\theta>\theta_0$ the authors propose taking observations as long as

$$A<\left[\frac{\partial}{\partial\theta}\prod_{i=1}^{m} f(x_i;\theta)\right]_{\theta=\theta_0}\Big/\prod_{i=1}^{m} f(x_i;\theta_0)<B,\quad m=1, 2, \cdots$$

and accepting or rejecting H according to which of these inequalities is violated first. Using Wald's methods, approximations to the power function and expected number of observations are obtained. An approximate optimum property of the test is derived from the fact that it is the limit of a sequence of sequential probability ratio tests.

E. L. Lehmann (Berkeley, Calif.).

✓***Hotelling, Harold. A generalized T test and measure of multivariate dispersion.** Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950, pp. 23–41. University of California Press, Berkeley and Los Angeles, 1951. \$11.00.

Generalized T-tests, valuable in various applications, were introduced by the author and used in problems of multivariate statistical quality control [chapter 3 of Selected techniques of statistical analysis . . . , McGraw-Hill, New York, 1947]. In the present paper the distributions of these bivariate T-statistics are derived. The author further considers the case where the ratios of variances and covariances of a normally distributed set of random variables are known, so that only a common factor γ, called the general degree of dispersion, remains to be estimated from the sample. A maximum likelihood estimate is given for this statistic, and its variance is determined. It is also shown that the proposed statistic has minimum variance in a certain class of unbiased statistics.

E. Lukacs (Washington, D. C.).

✓***Hoeffding, Wassily. "Optimum" nonparametric tests.** Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950, pp. 83–92. University of California Press, Berkeley and Los Angeles, 1951. \$11.00.

The author presents a survey of known as well as some new results of his own on non-parametric tests which are optimum in various specified senses. Most powerful rank order tests and some of the results of Lehmann and Stein

[Ann. Math. Statistics 20, 28–45 (1949); these Rev. 10, 723] on most powerful and most stringent tests are discussed and illustrated. A theory of optimum tests with respect to non-parametric alternatives based upon Wald's minimax principle is developed and a method for constructing such optimum tests if they exist is given.

R. P. Peterson (Seattle, Wash.).

Basu, D. A note on the power of the best critical region for increasing sample size. Sankhyā 11, 187–190 (1951).

By a simple application of the weak law of large numbers, the author shows that the most powerful test for a simple hypothesis against a simple alternative is consistent (i.e. the power of the test approaches unity as the sample size increases) provided the observed variables are independently and identically distributed. An example is given to show that, if this last condition is relaxed, this (and hence any) test is not necessarily consistent. *D. G. Chapman.*

√ ***Wald, Abraham. Basic ideas of a general theory of statistical decision rules.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 1, pp. 231–243. Amer. Math. Soc., Providence, R. I., 1952.

Exposition of the basic ideas of the author's theory of statistical decision functions [see also the author's book, Statistical decision functions, Wiley, New York, 1950; these Rev. 12, 193], with a statement of some results of the author and others which were recent at the time this address was given (September, 1950). *J. Wolfowitz* (Ithaca, N. Y.).

Wald, Abraham. Statistical decision functions. Ann. Math. Statistics 20, 165–205 (1949).

The content of this paper is contained in essentially the same form in the author's book of the same title [Wiley, N. Y., 1950; these Rev. 12, 193].

√ ***Robbins, Herbert. Asymptotically subminimax solutions of compound statistical decision problems.** Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950, pp. 131–148. University of California Press, Berkeley and Los Angeles, 1951. $11.00.

Let X_i, $i=1, \cdots, n$ be independent normal random variables with unit variance and $E(X_i)=\theta_i$ where each θ_i is ± 1. It is desired to classify each θ according to its sign, and the risk of a classification procedure is taken to be the expected number of errors. Then the minimax rule attributes to θ_i the sign of X_i and has constant risk r_n. If $p=n^{-1}\sum\theta_i$, the author exhibits another procedure which is "asymptotically subminimax" in that its risk is $<r_n$ whenever $|p-\frac{1}{2}|>c_n$ and only slightly exceeds r_n for $|p-\frac{1}{2}|<c_n$ where $c_n\to 0$ as $n\to\infty$.

E. L. Lehmann (Berkeley, Calif.).

√ ***Cochran, W. G. Improvement by means of selection.** Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950, pp. 449–470. University of California Press, Berkeley and Los Angeles, 1951. $11.00.

A "selection" of candidates may be said to be optimum if a fixed proportion α are chosen on the basis of measurements of n random variables $x_1, x_2, \cdots, x_n$ in such a way that the expected value of another variable y in the selected population is maximized. Using a method analogous to finding a best critical region for a test of a simple hypothesis against a simple alternative, the best rule is shown to be: select all candidates for which $\eta(x)\geqq k$, $\eta(x)$ being the regression of y on the x's and where k is chosen so that the proportion selected is α. Next a two-stage selection process is studied: here there is the important question of the choice of the level of selection at the two stages. Some numerical and general results are obtained in this direction. These optimum selections require a knowledge of the joint distribution of the $(n+1)$ variables. In practice, unknown parameters must be estimated and the definition of an optimum selection modified. By a numerical analysis, the author shows some justification for the frequently used practice of discarding some of the initial variables. Several unsolved problems in this and other directions are indicated.

D. G. Chapman (Seattle, Wash.).

Radhakrishna Rao, C. Statistical inference applied to classificatory problems. II. The problem of selecting individuals for various duties in a specified ratio. Sankhyā 11, 107–116 (1951).

Suppose that in some scale of measurement x (a vector-valued random variable) individuals suited for duty 1 (e.g., cooks) have a density function $f_1(x)$ while individuals suited for duty 2 (e.g., mathematicians) have a density function $f_2(x)$. If n individuals have been inducted randomly from these two populations and if n_1 ($\leqq n$) cooks are needed, how should they be selected on the basis of measurements x. The obvious solution, choosing the n_1 closest to the standards for cooks (more precisely the n_1 with the larger ratio $f_1(x)/f_2(x)$), minimizes the expected loss (!) (through misclassification). The same solution minimizes the maximum loss in the following situation: n_1 cooks and n_2 $(=n-n_1)$ mathematicians were originally inducted, but their records have been lost so they have to be resorted on the basis of measurements of x. The same problems are solved in the case where three groups are involved. [The examples are the reviewers.] *D. G. Chapman* (Seattle, Wash.).

***Reiersøl, Olav. Diferencialaj ekvacioj de specimenaraj distribuoj. [Differential equations of sampling distributions].** University Institute of Economics, Oslo, 1950. 30 pp. (Esperanto. English summary)

Given a set of n random variables $x_1, \cdots, x_n$ with joint probability density $f(x_1, \cdots, x_n)$. Given k statistics $y_1, \cdots, y_k$ $(k\leqq n)$ which are either rational functions of the x's or inverses of such functions. If f satisfies n linear partial differential equations with polynomial coefficients, the transformation from the x's to the y's induces k linear partial differential equations satisfied by the joint probability density $g(y_1, \cdots, y_k)$ of the statistics. From these equations g may be determined by adjoining special values, or the equations may be used to yield many useful properties of the distribution. The author illustrates the method by the non-central chi-square distribution, the variance ratio in which the numerator is non-central, and the correlation coefficient. *A. Blake* (Buffalo, N. Y.).

Reiersöl, Olav. Transformation from probability density to characteristic function by means of differential equations, and the inverse transformation. Portugaliae Math. 10, 71–80 (1951). (Esperanto. English summary)

Extending the investigation described in the preceding review, the author transforms a differential equation satisfied by a univariate probability density function to a differential equation for its characteristic function, and also makes the inverse transformation. He illustrates the method

with the non-central quadratic form and the correlation coefficient. *A. Blake* (Buffalo, N. Y.).

√ ***Hoel, Paul G. Confidence regions for linear regression.** Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950, pp. 75–81. University of California Press, Berkeley and Los Angeles, 1951. $11.00.

The author considers the construction of an optimum confidence region around the usual least squares estimate of a regression line, derivations from true regression being independently and normally distributed. A confidence band in the regression plane is constructed as the envelope of the single parameter family of lines obtained by requiring the two regression line parameters to obey a certain (undetermined) relationship. The area of this band weighted by the frequency function of the independent variate is minimised. The equation thus obtained is solved in the case for which this frequency function is a normal one, under suitable subsidiary conditions. It is found that this optimum region differs insignificantly from the simple region proposed by H. Working and H. Hotelling [J. Amer. Statist. Assoc. Suppl. **24**, 73–85 (1929)]. The generalisation of the method to the case of multivariate regression is indicated.

P. Whittle (Uppsala).

Neyman, Jerzy. Existence of consistent estimates of the directional parameter in a linear structural relation between two variables. Ann. Math. Statistics **22**, 497–512 (1951).

The problem is to estimate the direction parameter θ^* of an exact relation $\xi\cos\theta^*+\eta\sin\theta^*-p=0$ when the measurements X, Y of ξ, η are subject to error, $X=\xi+U$, $Y=\eta+V$; there is no prior knowledge of θ^*, and the distribution of U, V is assumed not to be exactly normal. With reference to the work of Reiersöl [Econometrica **18**, 375–389 (1950); these Rev. **12**, 347], a statistic θ is constructed after grouping the pairs of measurement into 8 groups, and θ is shown to be a consistent estimate of θ^*. *H. Wold* (Uppsala).

√ ***Wallis, W. Allen. Tolerance intervals for linear regression.** Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950, pp. 43–51. University of California Press, Berkeley and Los Angeles, 1951. $11.00.

The author considers the construction of tolerance intervals for deviates (for a particular value of the independent variate) from the least squares estimated regression line. The deviates from true regression are supposed normally and independently distributed with unknown variance. This last circumstance weakens the character of the tolerance interval, of which it can only be said that it contains, with a certain probability, more than a certain fraction of the population of pertinent observations. The author now uses the Wald-Wolfowitz approximation [Ann. Math. Statistics **17**, 208–215 (1946); these Rev. **8**, 478] to calculate the interval width, given the probability level and the population fraction. The method is worked out in detail and illustrated numerically on some economic material.

P. Whittle (Uppsala).

Guest, P. G. The fitting of polynomials by the method of weighted grouping. Ann. Math. Statistics **22**, 537–548 (1951).

This method of fitting polynomials for equally spaced observations replaces the orthogonal polynomial of the least squares method by a step function. Tables are presented for computing the b's in the polynomial

$$y_i=b_0+\sum_j b_j(x_i-\bar{x})^j,\quad i=1,2,\cdots,n,$$

for $j=1,2,\cdots,5$ and $n=7(1)55$. A worked out example is given for $n=25$. Approximate standard errors of the regression coefficients are also included. This method is about 90% as efficient as the least squares procedure and requires considerably less multiplying if only the b's are computed. However the ξ' method may be superior if estimated y's are required for all x's. Also the computing of successive reductions in the sum of squares is more complicated when the step function is used. One of the advantages of orthogonal polynomials is the ease of computing these successive reductions and hence in estimating the error variance. It would be useful to make a time-study of the step-function method, as was done by this reviewer for various orthogonal polynomial methods. *R. L. Anderson* (Raleigh, N. C.).

***Sherman, Jack. The use of orthogonal polynomials in curve fitting and regression analysis.** Proceedings, Industrial Computation Seminar, September 1950, pp. 78–80. International Business Machines Corp., New York, N. Y., 1951.

A short discussion is presented on the use of the ξ' polynomials, plus an example of a second degree polynomial for 10 equally spaced observations. The ensuing discussion brings out the usefulness of constructing special orthogonal polynomials when the observations are not equally spaced, if different dependent variates are to be used for the same set of points. *R. L. Anderson* (Raleigh, N. C.).

Blanc-Lapierre, André. Étude de quelques modèles statistiques relatifs à des problèmes de bruit de fond. Revue Sci. **89**, 139–150 (1951).

Mathematical Economics

Wold, Herman O. A. Statistical estimation of economic relationships. Econometrica **17** (Supplement), 1–22 (1949).

Expository paper. After some discussion of the inherent obstacles to statistical research in economics, the author turns to what he has called recursive systems of equations in economics. A recursive system is of the form

$$X_i(t)=C_i+\sum_{j=1}^{i-1}C_{ij}^{(0)}X_j(t)+\sum_{k=1}^{s}\sum_{j=1}^{n}C_{ij}^{(k)}X_j(t-K)+Y_i(t),$$

where $i=m+1,\cdots,n$, the variables $X_1,\cdots,X_m$ are exogenous, the rest endogenous, and $Y_i(t)$ is a residual. A recursive system is such that if all variables are known up to time $t-1$, the values for time t can be calculated in order. This, of course, reflects the subtriangular character of the matrix of the $C_{ij}^{(0)}$. The author stresses the causal interpretation of recursive equations. Recursive systems are always identified, in the sense of Haavelmo [cf. T. Koopmans, ed., Statistical inference in dynamic economic models, Wiley, New York, 1950, pp. 258–265; these Rev. **12**, 431]. In addition, if the residuals are uncorrelated, the single-equation regression estimates are maximum likelihood estimates, and consistent. Wold states also a theorem on the "resolving power" of recursive systems when the vector variable $X(t)$ is stationary (wide sense). *R. Solow* (Cambridge, Mass.).

√★**Arrow, Kenneth J. An extension of the basic theorems of classical welfare economics.** Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950, pp. 507–532. University of California Press, Berkeley and Los Angeles, 1951. $11.00.

The paper is concerned with the problem of distributing n commodities among m individuals. A distribution X consists of an assignment of amounts X_{ij} of the ith commodity to the jth individual. It is assumed that all commodities are continuously divisible and the vector $(X_{1j}, X_{2j}, \cdots, X_{nj}) = X_j$ is called the commodity bundle of the jth individual corresponding to the distribution X. The tastes of the jth individual are represented by a continuous utility function $U_j(X_j)$. The possible combinations of the n commodities which the economy is capable of producing form a set T which is assumed to be compact and convex as a subset of n-space. Calling a distribution realizable if the amounts of the commodities involved belong to T, a distribution X^* is called optimal if there exists no realizable X for which $U_j(X_j) \geqq U_j(X_j^*)$ for all j with strict inequality holding for at least one j.

Now for each set of prices $(p_1, p_2, \cdots, p_n)$ of the n commodities, it is clear that each individual will distribute his income so as to maximize $U_j(X_j)$ thus giving rise to some distribution X^*. The price vector $p = (p_1, p_2, \cdots, p_n)$ is said to equate supply and demand at this X^* if the total amount paid out for the combination of commodities of X^* is at least equal to the amount that would be paid out for any other realizable combination of commodities at these prices. It is proved that if the price vector p equates supply and demand at some X^*, then this X^* is optimal, and conversely, under suitable restrictions, for any optimal X^* there is a price vector p equating supply and demand at X^*.

D. Gale (Providence, R. I.).

Alexits, Georges. Théorie mathématique du trafic de marchandises sous le régime du capitalisme de monopole. Acta Math. Acad. Sci. Hungar. 1, 17–35 (1950). (French. Russian summary)

In using mathematics to analyze an economic problem, the author finds it necessary to point out the inadequacies of bourgeois economics and mathematical economics. On the assumptions that producers can utilize their surpluses only to expand production and that markets become saturated so as to serve as a drag on the growth of production, the author ends up ultimately with a set of differential equations of the type met with in the biological struggle for existence. Assuming certain shapes for the functions, he deduces the ultimate triumph of a single monopolist, whose ultimate growth is monotonic and whose wage payments must fall. If monopoly does not develop, then oscillations must rock any economic system obeying his postulated laws.

P. A. Samuelson (Cambridge, Mass.).

√★**Georgescu-Roegen, Nicholas. Relaxation phenomena in linear dynamic models.** Activity Analysis of Production and Allocation, pp. 116–131. Cowles Commission Monograph No. 13. John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1951. $4.50.

Special types of non-linear differential equations are treated, in which a developing system develops discontinuous changes in regime according to its present or immediately past history. A typical case is where the behavior of a vector x is governed by the matrix equation $dx/dt = ax$, where a itself becomes one of two sets of constants depending upon the magnitudes of dx/dt. The resulting system will be soluble for dx/dt in terms of x except possibly at the boundaries where the regimes shift, at which points the conditions on the system's development may become ambiguous or inconsistent, requiring us to go into the details of the system's immediate past. From any initial conditions x_0 the system will generate its own behavior, staying for finite times within the regimes 1 and 2, and shifting from 1 to 2 and 2 to 1 at times $t', t'', \cdots$. What the limiting form of behavior of the system will be, or its qualitative properties, appears to be a difficult and incompletely solved problem. Illustrations are given of dynamic input-output systems, and of national income models in which discontinuous hysteresis and relaxation phenomena are introduced.

P. A. Samuelson (Cambridge, Mass.).

Hénon, Robert. Offre et demande d'effort de l'ouvrier au travail. Econometrica 17 (Supplement), 287–294 (1949).

This study intends to show how the rate of output of a laborer is determined when the wage is a function of output. For an individual, the effort demand is specified by the real wage formula (1) $\rho = mx + n$, where ρ is the daily real wage, x is the rate of output expressed as the ratio between attained output and standard output, $m = d\rho/dx$ is the wage incentive, and n is a constant. $F(x)$ denotes the labor disutility function, such that $f(x) = dF(x)/dx$ is the final degree of disutility in the output. If the living standard for the wage ρ is defined by the utility function $\Omega(\rho)$, the final degree of utility in terms of wages is $\omega(\rho) = d\Omega(\rho)/d\rho$. For maximum satisfaction (2) $\omega(\rho)d\rho = kf(x)dx$. All of these functions are assumed to satisfy the usual conditions. (1) and (2) form the equilibrium system of an individual. If m is eliminated, we have (3) $(\rho - n)\omega(\rho) = kxf(x)$, which defines the set of equilibrium points (ρ, x) when m varies. There is a singular solution for each value of m if $n < \rho$. There follows a section on the definition and measurement of effort flexibility and income flexibility.

M. P. Stoltz (Providence, R. I.).

Roy, René. De la théorie des choix aux budgets de familles. Econometrica 17 (Supplement), 179–191 (1949).

This paper is concerned with the problem of aggregating individual demand functions. The first part is a summary of previous results by the author on individual demand [De l'utilité. Contribution à la théorie des choix, Actualités Sci. Ind., no. 930, Hermann, Paris, 1942; these Rev. 8, 217]. These results are contained in the system of n equations (1) $\varphi_i/q_i = -w$ $(i = 1, 2, \cdots, n)$, where $\varphi_i = \partial\Phi/\partial p_i$ and $w = \partial\Phi/\partial r$, and r = income. Define the total utility function by $U(\bar{Q}) = \mathfrak{U}$. In equilibrium $U(\bar{Q}) = \Phi(\bar{P}, r)$, where $\bar{Q}$ and $\bar{P}$ are quantity and price vectors. $\Phi(\bar{P}, r)$ = constant is the tangential equation of the indifference surface. The equations (1) are called the tangential equations of consumer equilibrium. This system permits the explicit expression of the quantity demanded as a function of p_i and r: (2) $q_i = -\varphi_i(\bar{P}, r)/w(\bar{P}, r)$. Putting $X_i = p_i/r$, the total utility function may be written: $U(\bar{Q}) = \Phi(\bar{P}, r) = \Psi(\bar{X})$. Hence, $q_i = \psi_i(\bar{X})/\sum X_i\psi_i(\bar{X})$, where $\psi_i = d\Psi/dX_i$.

Assuming the distribution of individual income to be fixed and independent of aggregate income, the law of collective demand for each commodity may be derived. The relative variation of individual demand for good i is given by

$$(3) \qquad \frac{dq_i}{q_i} = \sum_j \eta_{ij}\frac{dp_j}{p_j} + \epsilon_i\frac{dr_h}{r_h},$$

where the elasticity coefficients are defined by $\eta_{ij}=Eq_i/Ep_j$ and $\epsilon_i=Eq_i/Er$. For the collective demand:

$$(4)\qquad \frac{dQ_i}{Q_i}=\sum_k \beta_k \frac{dq_i}{q_i}=\sum_k\sum_j \beta_k\eta_{ij}\frac{dp_j}{p_j}+\sum_k \beta_k\epsilon_i\frac{dr_k}{r_k},$$

where Q_i is the aggregative demand and $\beta_i=q_i/Q_i$. The coefficient of collective demand for good i with respect to p_j is given by the weighted average $H_{ij}=\sum_k\beta_k\eta_{ij}$. Define, under the assumption that the income distribution is fixed, $E_i=\sum_k\beta_k\epsilon_i$; then

$$(5)\qquad \frac{dQ_i}{Q_i}=\sum_j H_{ij}\frac{dp_j}{p_j}+E_i\frac{dr}{r},$$

which may be written as: (6) $Q_i=Q_i(\bar{P}, r)$. Equation (6) is homogeneous of degree zero in the variables $\bar{P}$ and r.

It is shown that Slutsky's integrability conditions are satisfied in the special case where all income elasticities are unity. Under this same condition a collective utility function, equivalent to the real income of the community, may be said to exist. *M. P. Stoltz* (Providence, R. I.).

TOPOLOGY

✓**★Newman, M. H. A. Elements of the topology of plane sets of points.** 2nd ed. Cambridge, At the University Press, 1951. vii+214 pp. $4.75.

After a brief introduction to the algebra of sets, the author develops (in chapter II) the fundamental results on closed, open and compact sets in a metric space. The next chapter is devoted to continuous functions. The main body of purely set-theoretic topology culminates in chapter IV with characterizations of the arc and simple closed curve. Algebraic topology (based on methods peculiar to the author) begins in the following chapter in which proofs of the Jordan-Brouwer theorem and Brouwer's theorems on regional and dimensional invariance are given. The last two chapters are devoted mainly to connectivity properties of plane domains and contain the most difficult results in the text. Among other topics discussed in the course of the book are complete spaces, local connectivity, the boundary properties of plane regions and the theorem (Schoenflies) on extending a homeomorphism defined on the boundary of a Jordan domain. Applications are made to analysis. Fixed-point theorems are not mentioned. The choice of material is, as always, debatable. The emphasis on metric spaces seems unfortunate and the limited applicability of the grating theory is not an argument favoring its use. It ought to be said that the text is outstanding in one important aspect—the student will be convinced that topology is geometry.

A. D. Wallace (New Orleans, La.).

Nicolescu, Miron. On additive properties of sets and their applications. Acad. Repub. Pop. Romåne. Bul. Şti. A. **1**, 719–724 (1949). (Romanian. Russian and French summaries)

Let X be a set and P a property of subsets of X, written as $P(E)$ if E has the property P and as $\bar{P}(E)$ if E fails to have the property P. If $P(E)$ and $P(F)$ imply $P(E\cup F)$, then P is said to be additive. If $P(E)$ and $D\subset E$ imply $P(D)$, then P is hereditary. Let X be a topological space. A property P such that P(connected open set) whenever P(boundary of that set) is said to be of type D. A point $x\epsilon X$ is singular with respect to P if for every neighborhood U of x, $\bar{P}(U)$. The author asserts that in every separable metric space the set of points in a connected open set Δ which are singular with respect to an hereditary additive property of type D is either void or continuous and has non-void intersection with the boundary of Δ. (The term "continuous" is not defined, or mentioned in the proof.) The proof is wrong, as the author tacitly assumes that connected components of open sets are open, as well as committing a number of other errors. Various applications, to known theorems in analysis, are offered. *E. Hewitt.*

Obreanu, Filip. Open filters. Acad. Repub. Pop. Romåne. Bul. Şti. Ser. Mat. Fiz. Chim. **2**, 1–5 (1950). (Romanian. Russian and French summaries)

An open filter is a family of open subsets of a topological space closed under the formation of finite intersections, containing with an open set every open superset, and not containing the void set. An open ultra-filter is defined in the usual way. Five well-known and obvious facts concerning these objects are recounted. *E. Hewitt.*

Obreanu, Filip. Absolutely closed spaces. Acad. Repub. Pop. Romåne. Bul. Şti. Ser. Mat. Fiz. Chim. **2**, 21–25 (1950). (Romanian. Russian and French summaries)

The following properties of a topological space X are proved to be equivalent (for terminology, see the preceding review). 1) Every open filter on X has an adherent point. (This axiom is mis-stated in both the Russian and the French summaries but is given correctly in the paper itself.) 2) Every open ultra-filter on X is convergent. 3) Every open covering of X admits a finite subfamily whose closures cover X. 4) If $\{G_\xi\}_{\xi\epsilon Z}$ is a family of open subsets of X such that $\bigcap_{\xi\epsilon Z}G_\xi^-=0$, then there is a finite subfamily $G_{\xi_1},\cdots,G_{\xi_n}$ with void intersection. These properties are, of course, the absolute closure of Aleksandrov and Uryson [see, for example, M. H. Stone, Trans. Amer. Math. Soc. **41**, 375–481 (1937); M. Katětov, Časopis Pěst. Mat. Fys. **69**, 36–49 (1940); these Rev. **1**, 317]. The author proves also that a continuous image of an absolutely closed space in a Hausdorff space is absolutely closed, and that a product of spaces is absolutely closed if and only if every factor space is absolutely closed. *E. Hewitt* (Seattle, Wash.).

Obreanu, Filip. Espaces séparés minimaux. An. Acad. Repub. Pop. Romåne. Secţ. Şti. Mat. Fiz. Chim. Ser. A. **3**, 325–349 (1950). (Romanian. Russian and French summaries)

A Hausdorff space X is said to be minimal if every one-to-one continuous image Y of X which is also a Hausdorff space is homeomorphic to X. After reviewing a number of well-known facts about topological spaces, the author shows by examples that closed subspaces, continuous Hausdorff images, and retracts of minimal Hausdorff spaces need not be minimal. A product of Hausdorff spaces is minimal if and only if each factor-space is minimal. A minimal Hausdorff space is absolutely closed but not conversely, as an example shows. The author finally states that if a Hausdorff space X is the union of a countable number of absolutely closed subspaces F_n and is also the union of a countable number of nowhere dense sets M_n, then X has no one-to-one continuous image which is a compact Hausdorff space. The proof is incorrect, but becomes correct if one assumes that

each F_n is nowhere dense. The theorem as stated must be regarded as undecided. *E. Hewitt* (Seattle, Wash.).

Obreanu, Filip. Espaces localement absolument fermés. An. Acad. Repub. Pop. Române. Secţ. Şti. Mat. Fiz. Chim. Ser. A. 3, 375–394 (1950). (Romanian. Russian and French summaries)

A Hausdorff space X is said to be locally absolutely closed (LAC) if every point of X has an open neighborhood whose closure is absolutely closed (AC). The theory of such spaces is very like the theory of locally compact Hausdorff spaces, every standard structure theorem for the latter class of spaces being provable for the former with only verbal modifications. For example, a Cartesian product of Hausdorff spaces is LAC if and only if all factors are LAC and all but a finite number are AC. Also, a LAC space which is not AC can be imbedded (although not uniquely) in an AC space by the addition of a single point. Neighborhoods of the adjoined point can be taken as complements of absolutely closed sets in the original space. The real line R can be imbedded in a space $R\cup\{\omega\}$, where $U_n(\omega)=\{\omega\}\cup\{\text{all } x\varepsilon R$ such that $|x|>n$ and x is not an integer$\}$. As $R\cup\{\omega\}$ is absolutely closed but not compact, this example shows that the imbedding described above is not unique.

E. Hewitt (Seattle, Wash.).

Morita, Kiiti. A generalization of a theorem of C. Kuratowski concerning functional spaces. Sci. Rep. Tokyo Bunrika Daigaku. Sect. A. 4, 151–155 (1949).

Let R be a separable metric space and K a $(2n+1)$-dimensional cube in a $(2n+1)$-dimensional Euclidean space. Then the set of all continuous mappings of R into K is a complete metric space, where the metric is defined in the usual way. Denote this space by L. The author establishes the following theorem: Let $A_1, A_2, \cdots, A_m$ be a finite collection of closed sets in a separable metric space R of dimension not exceeding n. Then the set H of all continuous mappings f of R into K satisfying

$$f(A_1)\cdot f(A_2)\cdots f(A_m)=f(A_1A_2A_3\cdots A_m)$$

is a dense G_δ-set in L. If, in particular, the sets are disjoint, then H is open. The special case of this theorem in which $m=2$ and the set A_1A_2 is compact has been proven previously [see Kuratowski, Fund. Math. 30, 8–13 (1938)]. The theorem itself is of importance for discussing the problem of compactifications of topological spaces by the method of functional spaces. *D. W. Hall* (College Park, Md.).

Bing, R. H. A characterization of 3-space by partitionings. Trans. Amer. Math. Soc. 70, 15–27 (1951).

The author begins with the following combinatorial result: If C is a finite complex which is a 3-manifold without boundary, then C is a 3-sphere if the 3-simplexes of C may be ordered $c_1, c_2, \cdots, c_n$ such that c_j $(j=2, \cdots, n)$ intersects $c_1+c_2+\cdots+c_{j-1}$ in a connected set. Let S denote a metric, compact, locally connected, and connected space. A partitioning of S is a collection of mutually exclusive open sets whose sum is dense in S. A sequence $G_1, G_2, \cdots$ of partitionings is a decreasing sequence of partitionings if G_{i+1} is a refinement of G_i and the maximum of the diameters of the elements of G_i approaches 0 as i increases without limit. A partitioning is regular if each of its elements is the interior of the closure of this element. In terms of these notions the main theorem is formulated as follows. A necessary and sufficient condition that S be a 3-sphere is that one of its decreasing sequences of regular partitionings $G_1, G_2, \cdots$ have the following properties: (1) The boundary of each element of G_i is a simple surface. (2) If the boundaries of two elements of $\sum G_i$ intersect, this intersection is a 2-cell. (3) The intersection of the boundaries of three elements of G_i is one-dimensional at each of its points. (4) If g is an element of G_{i-1} ($g=S$ if $i=1$) the elements G_i in g may be ordered $g_1, g_2, \cdots, g_n$ so that $F(g_j)$ [$j=1$ (if $i>1$), $2, \cdots, n$] intersects $F(g)+F(g_1)+\cdots+F(g_{j-1})$ in a connected set.

The author concludes by presenting new characterizations of the simple closed curve and the simple closed surface. A necessary and sufficient condition that S be a simple closed curve is that one of its decreasing sequences of regular partitionings $G_1, G_2, \cdots$ have the following properties: (1) The boundary of each element of G_i is a pair of distinct points. (2) No three elements of G_i have a boundary point in common. A necessary and sufficient condition that S be a simple surface is that one of its decreasing sequences of regular partitionings $G_1, G_2, \cdots$ have the following properties: (1) The boundary of each element of G_i is a simple closed curve. (2) The intersection of the boundaries of 3 elements of G_i contains no arc. (3) If g is an element of G_{i-1} ($g=S$ if $i=1$) the elements of G_i in g may be ordered $g_1, g_2, \cdots, g_n$ so that $F(g_j)$ [$j=1$ (if $i>1$), $2, \cdots, n$] intersects $F(g)+F(g_1)+\cdots+F(g_{j-1})$ in a nondegenerate connected set. *W. W. S. Claytor* (Washington, D. C.).

Burgess, C. E. Continua and their complementary domains in the plane. Duke Math. J. 18, 901–917 (1951).

Conditions are given under which a continuum is either indecomposable or the sum of a unique pair of indecomposable continua. For example, this conclusion follows if the continuum M contains no domain, there exists a sequence of mutually exclusive continuous curves in M converging to M, and there exists a finite collection G of complementary domains of M such that there is a domain intersecting M and lying in the set consisting of M plus the sets of the collection G. Examples are given to show that the hypotheses of the main theorems cannot be weakened.

Please read the corresponding note on p. 1139 *H. M. Gehman* (Buffalo, N. Y.).

Krasnosel'skii, M. A. Two problems. Uspehi Matem. Nauk (N.S.) 6, no. 5(45), 162–165 (1951). (Russian)

The following problem is formulated. Let L denote a continuous map of the n-sphere S^n into itself, without fixed points. Let Φ be a continuous vector field on a closed $(n+1)$-cell T^{n+1} bounded by S^n with no zero-vectors on S^n, and let this field satisfy the condition that the vectors ΦLx and Lx are never similarly directed for any point x of S^n. Show that Φ has zero-vectors in the cell T^{n+1}. The author discusses the theorem of Borsuk on which this problem is patterned. It is shown that the problem above is related to one by Lusternik, only the two-dimensional case of which has so far been solved [Shkliarsky, Mat. Sbornik N.S. 16(58), 125–128 (1945); these Rev. 7, 136]. *L. Zippin.*

Moise, Edwin E. Affine structures in 3-manifolds. I. Polyhedral approximations of solids. Ann. of Math. (2) 54, 506–533 (1951).

This is the first of a series of papers. Although the objective of the series as a whole is not explicitly stated, the present results lead the reviewer to hope that some light may be cast upon the basic and extremely difficult problem of triangulating the topological 3-manifold. Let $S=f(B\times J)$ be the homeomorphic image in euclidean 3-space E^3 of the cartesian product of a compact connected 2-manifold B and the closed linear interval $[0, 1]$. It is proved that a poly-

hedral surface exists, homeomorphic to B and separating $f(B\times 0)$ from $f(B\times 1)$ in E^3. This principal result is deduced from a theorem on approximations by a complex K to a connected locally compact subset M^0 of a similarly restricted set M, where M^0 does not meet the boundary $\beta(M)$ and where the Vietoris Betti numbers $p_1(M)$ and $p_1(M^0)$ are equal. It is required that $p_1(K)\leqq p_1(M)$ and that $\beta(K)$ be a 2-manifold. Two other theorems, essential to subsequent papers, pertain to the existence of suitably restricted approximating complexes to certain members of nested triples of compact connected sets in E^3, such as $C_0\subset C_1\subset C_2$, where the closure of C_i-C_{i-1} $(i=1, 2)$ is homeomorphic to a cartesian product of the same form, $B\times J$, as above.
S. S. Cairns (Urbana, Ill.).

Uehara, Hiroshi. On a Hopf homotopy classification theorem. Nagoya Math. J. 3, 49–54 (1951).

Hopf's theorem classifying maps of an n-dimensional complex into the n-sphere has been generalized to the case of a compact Hausdorff space X of dimension $\leqq n$ and in this generalized form can be regarded as asserting the existence of an isomorphism of the nth cohomotopy group $\pi^n(X)$ onto the nth Čech cohomology group $H^n(X)$ with integral coefficients. In the present paper the author generalizes this result by replacing the n-sphere by a connected absolute neighborhood retract Y with $\pi_r(Y)=0$ for $r<n$. It is shown that the homotopy classes of X into Y can be made into a group, and that this group is isomorphic to the nth Čech cohomology group $H^n(X;\pi_r(Y))$ with coefficients in $\pi_r(Y)$. The proof is based on the fact that there is an $(n+1)$-dimensional complex R, consisting of a union of n-spheres and $(n+1)$-cells, and a mapping h: $R\rightarrow Y$ such that if A is any finite complex of dimension n then h induces a one-to-one correspondence between the homotopy classes of maps of A into R and the classes of maps of A into Y. This establishes the result if X is a complex, and the general result follows by a standard limiting process using the nerves of the coverings of X. *E. H. Spanier.*

Shimada, Nobuo, and Uehara, Hiroshi. On a homotopy classification of mappings of an $(n+1)$ dimensional complex into an arcwise connected topological space which is aspherical in dimensions less than n $(n>2)$. Nagoya Math. J. 3, 67–72 (1951).

Let X be a finite complex of dimension $\leqq n+1$, and Y an arc-wise connected topological space for which the homotopy groups $\pi_p(Y)$ are trivial for $p<n$. In this paper the author sketches the proof of a homotopy classification theorem for continuous maps of X into Y in case $n>2$ and $\pi_n(Y)$ is the direct sum of a countable number of cyclic groups. The author notes that his main result has already been announced (without proof) by M. M. Postnikov [Doklady Akad. Nauk SSSR (N.S.) 71, 1027–1028 (1950); these Rev. 11, 676]. The proof of the homotopy classification theorem is based on an extension theorem. The statement and proof of this extension theorem are direct generalizations of an extension theorem proved by Steenrod for the case of mappings into an n-sphere $(n>2)$ [Ann. of Math. (2) 48, 290–320 (1947); these Rev. 9, 154].

At the end of the paper, the following result is stated. Let X be an arbitrary finite complex, and let Y be an $(n-1)$-connected topological space. Denote the p-skeleton of X by X^p. Let f, g be two maps of X^{n+2} into Y which coincide on X^n. Denote the $(n+3)$-dimensional obstruction cocycles to the extension of f and g by $c^{n+3}(f)$ and $c^{n+3}(g)$ respectively, and the $(n+1)$-dimensional separation cocycle of f and g by $d^{n+1}=d^{n+1}(f, g)$ [cf. S. Eilenberg, Ann. of Math. (2) 41, 231–251 (1940); these Rev. 1, 222]. Then in case $n>2$, $c^{n+3}(f)-c^{n+3}(g)$ is cohomologous to $\mathrm{Sq}_{n-1}(d^{n+1})$. In case $n=2$, $c^5(f)-c^5(g)$ is cohomologous to $\mathrm{Sq}_1(d^3)+(c^2\cup d^3)$. Here $c^2=c^2(f)=c^2(g)$ is the characteristic cocycle, and in the definition of the cup product, $c^2\cup d^3$, the Whitehead product is used for the pairing of the coefficients. This is apparently the first published result on the "third" obstruction; however, no hint is given as to the method of proof.
W. S. Massey (Providence, R. I.).

✓**★Olum, Paul. The theory of obstructions.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 363–370. Amer. Math. Soc., Providence, R. I., 1952.

A review of the rather classical obstruction theory is given for the homotopy of the mappings of X into Y leaving the image of a point x_0 of X fixed at a point y_0 of Y. In the later part of the paper, the author announces some new results. Namely, he considers two connected triangulable n-manifolds M^n and Q^n such that $\pi_r(Q^n)=0$ for $1<r<n$ and studies the mappings of M^n into Q^n which carry a given point x_0 of M^n into a given point y_0 of Q^n. A few theorems are stated concerning whether or not two such mappings are homotopic relative to x_0. As an application, the author gives a homotopy-type classification of the higher dimensional lens spaces as follows: Two $(2n+1)$-dimensional lens spaces $L(m; q_1, \cdots, q_n)$ and $L'(m'; q_1', \cdots, q_n')$ are of the same homotopy-type if and only if $m=m'$ and $q_1q_2\cdots q_n$ is congruent with $\pm k^2q_1'q_2'\cdots q_n'$ modulo m for some k relatively prime to m. *S. T. Hu* (Princeton, N. J.).

✓**★Whitehead, George W. Homotopy groups of spheres.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 358–362. Amer. Math. Soc., Providence, R. I., 1952.

The Freudenthal suspension, the Whitehead product, the author's generalization of the Hopf invariant, and his composition $[\pi_n(S^p), \pi_p(x)]\epsilon\pi_n(x)$ are described, their main properties and interrelations stated, and their consequences for the homotopy groups of spheres sketched. Also explained is a method for obtaining the non-zero element of $\pi_{n+2}(S^n)$ from the rotation group of S^{n-1}. *J. Dugundji.*

✓**★Massey, W. S. Homotopy groups of triads.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 371–382. Amer. Math. Soc., Providence, R. I., 1952.

The triad homotopy groups, the "homotopy groups of a covering" and generalized Whitehead products are described, and their main properties stated. If X^* is a space obtained from X by adjunction of the n-cell $\mathcal{E}^n$, and if $(X, \dot{\mathcal{E}}^n)$ is m-connected $(m\geqq 1)$, this machinery leads to a description of the first non-trivial triad group $\pi_{m+n}(X^*; \mathcal{E}^n, X)$ in case X, $\dot{\mathcal{E}}^n$ are compact ANR, $\pi_1(X)=0$, and the identification map sends S^{n-1} homeomorphically onto $\dot{\mathcal{E}}^n$. Applications of this result are also given. *J. Dugundji.*

Blakers, A. L., and Massey, W. S. The homotopy groups of a triad. II. Ann. of Math. (2) 55, 192–201 (1952).

Let $(X; A, B)$ be a triad with (a) $A, B, A\cap B$ arcwise connected and (b) $X=\mathrm{Int}\,A\cup\mathrm{Int}\,B$. The authors show that if $(A, A\cap B)$ is m-connected and $(B, A\cap B)$ is n-connected, $m\geqq n>1$, then the triad $(X; A, B)$ is $(m+n)$-

connected; suitable conditions on the couples $(A, A\cap B)$ and $(B, A\cap B)$ yield this theorem also for $m\geqq n\geqq 1$. It is then shown that (b) can be weakened: A and B closed, and $A\cap B$ a deformation retract (remaining pointwise fixed during the deformation) of an open set in B (or A). Applications are given to the homotopy groups of CW complexes with a closed subcomplex shrunk to a point, to the injection $\pi_n(A, A\cap B)\to\pi_n(X, B)$, and to the Eilenberg-MacLane "suspension" [same Ann. **46**, 480–509 (1945); these Rev. **7**, 137]. *J. Dugundji* (Princeton, N. J.).

✓★**Hurewicz, W. Homotopy and homology.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 344–349. Amer. Math. Soc., Providence, R. I., 1952.

In this expository address, the author gives a resumé of the various relations between homotopy and homology which have been established since L. E. J. Brouwer demonstrated in 1912 that two continuous mappings of a 2-sphere into itself are homotopic if and only if they have the same degree. Also included are brief intuitive descriptions of some of the main concepts of modern homotopy theory, including the following: absolute and relative homotopy groups, the natural homomorphism of the homotopy groups of a space into the homology groups of the space, obstructions to the extension of a continuous map, triad homotopy groups, and the classification of spaces according to homotopy type.

W. S. Massey (Providence, R. I.).

✓★**Spanier, E. H. Homology theory of fiber bundles.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, Vol. 2, pp. 390–396. Amer. Math. Soc., Providence, R. I., 1952.

Let B be a fibre bundle with base space X and fibre F. The author discusses the problem of determining relations between the homology (or cohomology) groups of B, X, and F. The discussion is restricted to the case in which X is a finite cell complex because, in the words of the author, "the situation is simpler than the general case, and computations can be carried out more easily." The main technique used is a study of the homology properties of the inverse images of the various skeletons of the base space under the projection. This technique was utilized by Chern and the author in a previous paper [Proc. Nat. Acad. Sci. U. S. A. **36**, 248–255 (1950); these Rev. **12**, 42] to study the case in which the fibre F is a sphere. Most of the results of this previous paper are summarized in the present discussion. There is also some discussion of the general case in which the fibre is not a sphere, with particular emphasis on the algebraic machinery involved. The author states several applications of his methods. Among these is the theorem that if the fibre is totally nonhomologous to zero, then the cohomology groups of the bundle are isomorphic to the corresponding groups of the product space of the base space by the fibre. *W. S. Massey.*

✓★**Hirsch, Guy C. Homology invariants and fibre bundles.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 383–389. Amer. Math. Soc., Providence, R. I., 1952.

The author summarizes the results known about the homology properties of fibre bundles. In particular, he considers the problem of determining the homology of the bundle when the homology of the base and the fibre are known. The author attacks this problem by a method previously considered by him [C. R. Acad. Sci. Paris **227**, 1328–1330 (1948); these Rev. **10**, 558] and shows that the (additive and part of the multiplicative) homology properties of the bundle can be determined from knowledge of the characteristic isomorphism which connects the homology of the fibre with the homology of the base. To obtain the complete multiplicative structure of the bundle, new invariants of the fibre bundle are needed, and, conversely, the multiplicative structure determines these invariants. For example, given a sphere bundle with structure group the orthogonal or unitary group there is an auxiliary bundle whose multiplicative structure determines the characteristic classes of the given sphere bundle. *E. H. Spanier.*

Koseki, Ken-iti. Über die Homöomorphien der offenen Flächen. II. Mem. Coll. Sci. Univ. Kyoto Ser. A. Math. **26**, 113–137 (1951).

In the first part [same Mem. **26**, 75–94 (1951); these Rev. **12**, 729] the author defined a (topologically invariant) subgroup, B_1, of the one-dimensional homology group, mod 2, of an orientable open surface, K. In the present paper, the elements, $b_1, \cdots, b_n$, of B_1, are said to have no common point if we may choose simple closed polygons $P_1, \cdots, P_n$, on some simplicial subdivision of K, representing $b_1, \cdots, b_n$, no two of which have a point in common. If P_1 and P_2 are separated by P_3, then b_1 and b_2 are said to be separated by b_3, and this property of elements of B_1 is shown to be independent of the choice of representing polygons. Similarly, the author defines the property "b_2 is in the side of finite (infinite) genus with respect to b_1" to mean that the region determined by P_1 which contains P_2 is of finite (infinite) genus.

It is shown that a basis for B_1 may be chosen consisting of elements without a common point. Let us call this an admissible basis. Then the main theorem asserts that two orientable open surfaces K and K' are homeomorphic if and only if (1) K and K' have the same genus; (2) a one-to-one correspondence $b_n\leftrightarrow b_n'$ may be set up between the elements of an admissible basis for B_1 and the elements of an admissible basis for B_1', such that b_k separates b_i and b_j if and only if b_k' separates b_i' and b_j'; (3) if a_1, a_2 are arbitrary distinct non-zero elements of B_1 and a_1', a_2' their images under the isomorphism $B_1\to B_1'$ generated by the one-to-one correspondence of (2), then a_2 is in the side of finite genus with respect to a_1, if and only if a_2' is in the side of finite genus with respect to a_1'. A slight modification is required in (3), of course, if B_1 has a unique non-zero element.

P. J. Hilton (Manchester).

Miščenko, E. F. On the homology theory of non-closed sets. Mat. Sbornik N.S. **29(71)**, 587–592 (1951). (Russian)

For an arbitrary subset C of the euclidean n-sphere S^n, $\Delta_c{}^pC$ denotes the group of true p-cycles with compact carriers in C, modulo those homologous to 0 on compact subsets of C. Aleksandrov has proved [Mat. Sbornik N.S. **21(63)**, 161–232 (1947); these Rev. **9**, 456] that if A is the topological image in S^n of a "stripped" polyhedron and $B=S^n-A$ then $\Delta_c{}^pA$ and $\Delta_c{}^qB$, $p+q=n-1$, are dual, where the coefficient group $\mathfrak{A}$ for A is discrete and the coefficient group $\mathfrak{B}$ for B is compact and $\mathfrak{A}$ and $\mathfrak{B}$ are dual. The author proves the same conclusion for B a curved stripped polyhedron and $A=S^n-B$, thus yielding a result symmetric in A and B. To prove this, the author shows that if B is a curved stripped polyhedron, then A is a homology retract

over any discrete coefficient group. He then applies Aleksandrov's result that if A is a homology p-retract over $\mathfrak{A}$, then $\Delta_\epsilon^p A$ and $\Delta_\epsilon^q B$ are dual. A shorter proof of this result of Aleksandrov is also given, as well as some other contributions to the theory of Aleksandrov. *E. E. Floyd.*

Sitnikov, K. On homological girdling of compacta in Euclidean space. Doklady Akad. Nauk SSSR (N.S.) **81**, 153–156 (1951). (Russian)

This is a generalization of the author's earlier work [same Doklady (N.S.) **66**, 1059–1069 (1949); these Rev. **11**, 45]. It contains as a special case the homological characterization of closed r-dimensional subsets of euclidean n-space R_n in P. Alexandroff's note [Math. Ann. **106**, 161–238 (1932)].

Let x be a simplicial chain in R_n. For $p\geqq 0$, let $\alpha^p x$ denote the lower bound of all $\epsilon>0$ such that there is an ϵ-displacement of the vertices of x following which all simplexes of x collapse to dimension at most p. Let τx be the lower bound of all ϵ such that there exists an ϵ-displacement of vertices which takes x into a zero-chain. The author proves the following theorem: Let F be an r-dimensional compactum in R_n, and let $\Gamma=R_n-F$. There exists a real number $\gamma>0$ such that for every $k=1, \cdots, r+1$, and arbitrary $\epsilon>0$ there is an $(n-k)$-dimensional cycle v (called a girdle about F, of dimension $n-k$) which is bounding in Γ for $k>1$, and which satisfies these conditions: a) $\alpha^{r-k+1}v<\epsilon$, b) $\tau v<\epsilon$, and furthermore has the following properties: 1) for every cycle w homologous to v in a γ-neighborhood of v (relative to Γ), $\alpha^{r-k}w>\gamma$; 2) for every chain x which can be displaced onto v in G, $\alpha^{r-k+1}x>\gamma$.

On the other hand, if $s>r$ and $k=1, \cdots, s+1$, then, for an arbitrary $\gamma>0$, every $(n-k)$-dimensional cycle z in Γ for which $\tau z<\gamma$ is homologous, in its own γ-neighborhood relative to Γ, to some cycle x' with arbitrary small $\alpha^{s-k}x'$. Further, if $s>r$ and $k=2, 3, \cdots, s+1$, then for arbitrary $\gamma>0$ every $(n-k)$-dimensional cycle z which is bounding in Γ and for which $\alpha^{s-k+1}z<\gamma$ (and $\tau z<\gamma$, for $s=n-1$) bounds in Γ a chain x for which $\alpha^{s-k+1}x<\gamma$. *L. Zippin.*

Bassi, Achille. Dualità nelle varietà con contorno e varietà contorno completo di altre. Revista Científica **2**, 33–35 (1951).

The author announces a sharpening of the Lefschetz duality theorem for manifolds with regular boundaries which implies conditions on the homology groups of the boundary not satisfied by absolute manifolds generally. This makes it possible to list various classes of manifolds which cannot be boundaries, e.g. the real projective spaces of dimension $n\neq 4p+1$ (the case $n=4p+1$ remains open.) *P. A. Smith* (New York, N. Y.).

GEOMETRY

Court, Nathan Altshiller. Sur les triangles homologiques. Mathesis **60**, 233–238 (1951).

Thébault, Victor. Perspective and orthologic triangles and tetrahedrons. Amer. Math. Monthly **59**, 24–28 (1952).

Blanchard, R., et Thébault, V. Sur la cubique de Mac Cay. Mathesis **60**, 244–248 (1951).

Monseau, M. Produit des distances de deux points conjugués isogonaux à une droite quelconque du plan du triangle. Mathesis **60**, 256–263 (1951).

Thomissen, F., und Tromp, G. Über einige Konstruktionen, die auf den Sätzen von Pascal und Sturm beruhen. Elemente der Math. **7**, 5–8 (1952).

Sanielevici, S. Remark on the "Regula mnemonica" of Napier. Acad. Repub. Pop. Române. Bul. Şti. Ser. Mat. Fiz. Chim. **2**, 541–544 (1950). (Romanian. Russian and French summaries)

L'auteur montre que la "règle mnémonique" de Neper découle d'une manière naturelle de la correspondance établie par Lobatchevsky entre un triangle sphérique rectangle et un triangle rectiligne rectangle du plan "hyperbolique". *Author's summary.*

Hohenberg, Fritz. Die Brennpunktseigenschaften der Kegelschnitte im komplexen Gebiet. Elemente der Math. **6**, 121–129 (1951).

Coşniţă, Cezar. Sur la transformation quadratique. Acad. Répub. Pop. Roum. Bull. Sect. Sci. **30**, 391–395 (1948).

Maruyama, Takaharu. Some properties on geometry in complex space. I. J. Sci. Gakugei Fac. Tokushima Univ. **1**, 31–35 (1950).

A few elementary properties of the planes of the real 4-dimensional space corresponding to the straight lines of the complex plane. *P. Abellanas* (Madrid).

Bilo, J. Conditions for the equivalence of point-sets in quaternion projective geometry. Simon Stevin **28**, 140–145 (1951).

Continuing his Onderzoekingen betreffende de meetkundige grondslagen van de projectieve quaternionenmeetkunde [Vanderlinden, Brussels, 1949; these Rev. **11**, 612] the author proves that two sets of five collinear points, $ABCDE$ and $A'B'C'D'E'$, are equivalent if and only if

$$ABCD\,\overline{\wedge}\,A'B'C'D', \quad ABCE\,\overline{\wedge}\,A'B'C'E',$$

and

$$ABDE\,\overline{\wedge}\,A'B'D'E'.$$

H. S. M. Coxeter (Toronto, Ont.).

Castrucci, Benedito. Calculation of the order of the group of homographies of the n-dimensional space over a field of order $q=p^n$. Bol. Soc. Mat. São Paulo **3**, no. 1–2 (1948), 17–20 (1951). (Portuguese)

L'auteur effectue le calcul indiqué dans le titre, sans se servir des équations de l'homographie, et en utilisant seulement le résultat connu de géométrie projective finie N-dimensionnelle suivant lequel, sur un corps d'ordre $q=p^n$, le nombre de points de l'espace P_N est $(q^{N+1}-1)/(q-1)$. Il obtient ainsi la formule

$$H=\frac{1}{q-1}\prod_{i=0}^{N}(q^{N+1}-q^i),$$

donnant le nombre total H des homographies considérées. *P. Vincensini* (Marseille).

✱**Radojčić, M. Sur les points de vue qui dominent la géométrie.** Premier Congrès des Mathématiciens et Physiciens de la R.P.F.Y., 1949. Vol. II, Communications et Exposés Scientifiques, pp. 37–48. Naučna Knjiga, Belgrade, 1951. (Serbo-Croatian. French summary)

✱**Segre, Beniamino. Géométrie mathématique et géométrie physique.** Congrès International de Philosophie des Sciences, Paris, 1949, vol. III, Philosophie Mathématique, Mécanique, pp. 37–46. Actualités Sci. Ind., no. 1137. Hermann & Cie., Paris, 1951.

Algebraic Geometry

Godeaux, Lucien. Sur les points d'Eckardt d'une surface algébrique. Mathesis 60, 253–256 (1951).

Godeaux, Lucien. Remarques sur les surfaces du quatrième ordre contenant une sextique de genre trois. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 37, 419–426 (1951).

Room [Proc. London Math. Soc. (2) **51**, 348–400 (1950); these Rev. 12, 125, 126] studied three determinantal quartic surfaces having a birational transformation between each pair such that the resultant of these three transformations is a non-periodic transformation of either surface into itself. Each surface has on it two linear systems of sextics of genus 3, residual with respect to cubic sections, whose projective models are the other two surfaces.

The present paper points out that the determinantal quartic can be defined precisely as one having on it a sextic of genus 3, and from this point of view was studied by the author a good many years ago [Bull. Int. Acad. Polon. Sci. Cl. Sci. Math. Nat. Ser. A. Sci. Math. **1913**, 529–547]; it is also stated that as the three surfaces are birationally equivalent, they have the same moduli and are thus projectively equivalent; this however seems wrong to the reviewer, since if one of them is the Weddle surface, another is the Kummer surface, and these are not projectively equivalent since they have not the same number of nodes. The main observation however is that in the general case, where the plane sections $|C_0|$ and one system $|C_1|$ of sextics form a base for linear systems on the surface, the general system $|\lambda_0C_0+\lambda_1C_1|$ has genus $\lambda_0^2+3\lambda_0\lambda_1+\lambda_1^2$, and grade 4 times this expression; and the transformations of the three surfaces into themselves, or each other, are just all the linear transformations on $|C_0|$, $|C_1|$, of unit determinant, and leaving invariant this quadratic form. If T_1 interchanges $|C_0|$ with $|C_1|$, and T_2 interchanges $|C_0|$ with $|C_2|=|3C_0-C_1|$, these operations are all those of the forms $(T_1T_2)^k$, $(T_1T_2)^kT_2$, of which the latter are all involutory. *P. Du Val.*

Babbage, D. W. An algebro-geometric interpretation of the associated forms of a binary form. Proc. London Math. Soc. (3) 1, 170–177 (1951).

If $P_1, P_2, \cdots, P_n$ are n points of which no $n-1$ are coprimal in $[n-2]$, and if the set P_i is determined on a particular twisted cubic by an equation $f=0$ (where f is a binary form), particular covariant sets of P_i are given by the vanishing of the "Schwesterformen".

M. Piazzolla Beloch (Ferrara).

Derwidué, L. Décomposition des transformations birationnelles en produits de transformations élémentaires. Math. Ann. 124, 65–76 (1951).

The present paper is closely related to the author's recent paper [Math. Ann. **123**, 302–330 (1951); these Rev. **13**, 67]. The problem which the author studies now is again a fundamental one and very difficult, as appears clearly from the title of the present paper. His essential tools are two theorems which appeared in the cited paper of the author: a theorem on first polars and the "raisonnement fondamental". Since the first of these two theorems is false, while the second has not been fully established [see the cited review], the author's conclusions cannot be accepted for the present.

O. Zariski (Cambridge, Mass.).

Segre, B. On the inflexional curve of an algebraic surface in S_4. Quart. J. Math., Oxford Ser. (2) 2, 216–220 (1951).

Let P be a general point of an irreducible algebraic surface F in S_4. With the exception of a certain class of surfaces, called parabolic, the pairs of tangents at P to the sections of F by the pencil of tangent primes at P form an involution. The locus of points for which the double lines of this involution coincide is the inflexional curve Γ of F. For a Segre surface, the intersection of two quadric primals, Γ consists of sixteen lines, and is the complete intersection of F and a quartic primal. This fact is generalized as follows: If F is the intersection of two general primals of orders $m, n>1$, then Γ is the complete intersection of F with a primal of order $6m+6n-20$. If F is non-parabolic and free from multiple curves, then $\Gamma\equiv 10C+6K$, where C is a prime section and K a (virtual) canonical curve. If F is of order ν and rank ρ then Γ is of order $6\rho-8\nu$. *R. J. Walker.*

Segre, Beniamino. Alcune questioni algebrico-differenziali. Portugaliae Math. 10, 29–36 (1951).

Ce travail est lié étroitement avec une autre recherche de l'auteur [voir l'analyse ci-dessus], quoique le problème se pose en forme complètement projective. Si, en effet, la surface de S_4 est birationnellement représentable sur un plan, ses sections hyperplanes seront représentées par un système lineaire ∞^4 de courbes dont les systèmes partiels correspondants aux sections déterminées par les hyperplans qui passent pour une même tangente d'inflexion sont des gerbes ayant un contact du deuxième ordre fixe.

L'auteur pose d'abord le problème général de la détermination et de l'étude de la courbe J_d, lieu des points qui, par rapport à un système linéaire ϕ_{2d} de courbes algébriques de dimension $2d$, sont les points-bases de systèmes partiels de dimension d, ayant en ces points un contact fixe d'ordre d. Pour $d=1$, J_1 est la courbe jacobienne de ϕ_2; pour $d=2$, J_2 correspond, dans la représentation hyperspatielle, à la courbe d'inflexion de la surface représentative; avec une élégante considération algébrique, l'auteur donne l'équation de J_2, dont il calcule l'ordre $10n-18$ (n=ordre des courbes ϕ) et la multiplicité dans les points-bases de ϕ_{2d}. Dans le §IV, à propos de la considération des systèmes ϕ_4 dont la J_2 résulte indéterminée, on ajoute une observation générale sur les lignes asymptotiques des surfaces de S_m ($m\geqq 4$) qui satisfont une équation de Laplace de type parabolique. Dans le §V est contenu un aperçu sur d'autres problèmes apparentés avec le sujet de la note.

B. Levi (Rosario).

Oleĭnik, O. A. On the topology of real algebraic curves on algebraic surfaces. Mat. Sbornik N.S. **29**(71), 133–156 (1951). (Russian)

The author gives the proof of a result which he has announced in a previous note [Doklady Akad. Nauk SSSR (N.S.) **70**, 13–14 (1950); these Rev. **11**, 613]. Since there seems to be a slight discrepancy between the announced result and the result which is actually proved (see Theorem 2, p. 152), the latter will be reproduced here in full. Let Γ: $F(x, y, z)=0$ and γ: $f(x, y, z)=0$ be two non-singular real surfaces, of order p and q respectively, in the real projective 3-space, and let these surfaces intersect along a real curve K. It is assumed that every point of K is a simple intersection of Γ and γ (whence K is a non-singular curve; in particular, the intersection $\Gamma\cap\gamma$ contains no isolated real points). Let M_c, c real, denote the closure (in the projective space) of the set of points of Γ which are at finite distance and at which $f(x, y, z)\geqq c$. It will be noted that if q is odd, then the boundary of M_0 will consist not only of K but also of the curve at infinity on Γ. The main result is the following estimate of the absolute value of the Euler-Poincaré characteristic $E(M_0)$:

$$|E(M_0)|\leqq\tfrac{1}{3}p^3+\tfrac{2}{3}pq^2+\tfrac{1}{4}p^2q-p^2-pq+\tfrac{7}{6}p+\tfrac{1}{2}|E(\Gamma)|,\quad\text{if } q \text{ is even;}$$

$$|E(M_0)|\leqq\tfrac{1}{3}p^3+\tfrac{2}{3}pq^2+\tfrac{1}{4}p^2q-\tfrac{3}{4}p^2-\tfrac{3}{4}pq+\tfrac{13}{12}p+\tfrac{1}{2}|E(\Gamma)-\sigma|,\quad\text{if } q \text{ is odd,}$$

where σ is the number of intersections of K with the plane at infinity (this integer σ does not seem to be present in the result which was announced in the cited note). It is shown that in the case of quadric surfaces ($p=2$) these estimates are the best possible. The methods used in the present proof are variational in nature and are similar to those used previously by I. Petrovskiĭ and the author in their joint paper on the topology of real algebraic surfaces [Izvestiya Akad. Nauk SSSR. Ser. Mat. **13**, 389–402 (1949); these Rev. **11**, 613]. In the present case, the critical values of the parameter c are the values of the function f at the zeros of the differential df, f being regarded as a function on the surface Γ.

O. Zariski (Cambridge, Mass.).

Oleĭnik, O. A. Estimates of the Betti numbers of real algebraic hypersurfaces. Mat. Sbornik N.S. **28**(70), 635–640 (1951). (Russian)

Let Γ be a real non-singular algebraic hypersurface $F(x_1, x_2, \cdots, x_m)=0$, of order n, in the real projective m-space. For any complex K let $\sigma_1(K)$ and $\sigma_2(K)$ denote the sum of the Betti numbers of K of even or odd dimensions respectively, and let $\sigma(K)=\sigma_1(K)+\sigma_2(K)$. Using the results of a joint paper with Petrovskiĭ [Izvestiya Akad. Nauk SSSR. Ser. Mat. **13**, 389–402 (1949); these Rev. **11**, 613], the author now obtains upper bounds for $\sigma(K)$, $\sigma_1(K)$ and $\sigma_2(K)$, for $K=\Gamma$ and $K=M_0$, where M_0 has the same meaning as in the preceding review. These upper bounds are given in terms of m and n, and they also depend on whether m (or n) is even or odd. In view of the large number of cases involved, only the upper bounds for $\sigma(\Gamma)$, $\sigma_i(\Gamma)$, $i=1, 2$, will be given here (the last two bounds are independent of i). Let $S(m, n)$ denote the number of terms of the polynomial $\prod_{i=1}^{m}(x_i^{n-1}-1)/(x_i-1)$ which are of degree $\leqq[(mn-2m-n)/2]$ ($[a]$ stands for the integral part of a). For the purposes of this review it will be convenient to introduce a symbol $S'(m, n)$, defined as follows: $S'(m, n)=S(m, n)$ if m is odd, and $S'(m, n)=(n-1)^m/2$ if m is even. Then

$$\sigma_i(\Gamma)\leqq(n-1)^m-S'(m, n)+[(m+1)/2] \text{ if } n \text{ is odd,}$$

$$\sigma_i(\Gamma)\leqq(n-1)^m-S'(m, n)+\{(n-1)^m-(n-1)\}/2(n-2)+m+m(m-1)/4 \text{ if } n \text{ is even.}$$

If in these expressions we replace $S'(m, n)$ by $(n-1)^m/2$ (whether m is even or odd) and multiply the result by 2, we obtain the author's upper bound for $\sigma(\Gamma)$ (except that when n is odd the integer $2[(m+1)/2]$ should be replaced by m). For $m=3$, the above results yield upper bounds for the number of connected components of a real algebraic surface, of order n, in S_3. As in the previous work of the author, the proofs are based on variational considerations and make use of the duality theorems of Pontryagin.

O. Zariski (Cambridge, Mass.).

Roth, Leonard. Sugli invarianti d'una varietà algebrica a tre dimensioni. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **10**, 468–472 (1951).

The author's object in this paper is to indicate a method by which the relations connecting the numerical invariants of an algebraic V_3 can be established in a systematic way, and without recourse to unproved assumptions. The V_3 is supposed to have elementary singularities of the type appropriate to its ambient space, and the idea of the author is to compute the invariants of V_3 as functions of its projective characters, and so to obtain the linear relations which exist between them. The formulae quoted have all been obtained previously.

J. A. Todd (Cambridge, England).

Semple, J. G. A property of projected Segre varieties. Quart. J. Math., Oxford Ser. (2) **2**, 212–215 (1951).

The Segre variety $V(r)$ of dimension $2r$ in S_{r^2+2r} is defined parametrically by $X_{ij}=x_iu_j$, $i, j=0, \cdots, r$. The points of $V(r)$ are thus in one-to-one correspondence with the point-prime pairs (p, π) of S_r, and so are the points of $\Omega(r)$, the projection of $V(r)$ from the point $X_{ij}=\delta_{ij}$ into the prime $\sum X_{ii}=0$. Calling two points (p, π) and (p', π') of $\Omega(r)$ "related" if p is on π' and p' on π, the points related to a given point P form a $V(r-1)$ on $\Omega(r)$, and P is the residual intersection of $\Omega(r)$ with the S_{r^2-1} containing $V(r-1)$. The "coincidence locus" E, of self-related points, is the intersection of $V(r)$ and $\sum X_{ii}=0$. The set of $r+1$ pairwise related points corresponding to the vertices and opposite faces of an r-simplex in S are just the intersections of an $(r+1)$-secant S_{r-1} of $\Omega(r)$, and conversely. Applying these results to the special case $r=2$ one obtains properties of $\Omega(2)$ analogous to those of a projected Veronese surface in S_4. In particular, $\Omega(2)$ is uniquely determined by its coincidence locus E.

R. J. Walker (Ithaca, N. Y.).

Northcott, D. G. Some properties of analytically irreducible geometric quotient rings. Proc. Cambridge Philos. Soc. **47**, 662–667 (1951).

Suppose that $\mathrm{P}=k[\xi_1, \cdots, \xi_n]$ is a finite integral domain over the field k with the quotient field F. Then each prime ideal $\mathfrak{p}$ of P determines in the terminology of the author a geometric quotient ring Q, that is, the quotient ring $R(\mathrm{P}; \mathrm{P}-\mathfrak{p})$ of P with respect to the multiplicatively closed system $\mathrm{P}-\mathfrak{p}$. Combining results of Zariski and F. K. Schmidt on the normalization theorem and integral dependence it follows that the integral closure P^* of P in a given

finite algebraic extension E of F is again a finite integral domain over k. Next it is shown that the integral closure Λ of Q in E, which is identified with the quotient ring $R(\mathfrak{P}^*; \mathfrak{P}-\mathfrak{p})$, is a semi-local ring in the sense of Chevalley. The maximal ideals $\mathfrak{m}_i^*$, $1\leqq i\leqq h$, of Λ are those prime ideals of Λ which contract to the ideal of non-units of Q, and their corresponding quotient rings Q_i^* are again geometric quotient rings which equal the integrally closed rings $R(\mathfrak{P}^*; \mathfrak{P}^*-(\mathfrak{m}_i^*\cap \mathfrak{P}^*))$. The author shows that

$$\Lambda=Q_1^*\cap Q_2^*\cap\cdots\cap Q_h.$$

In order to investigate further the concept of integral dependence it is assumed that Q is analytically irreducible in the sense of Zariski, that is, its completion $\bar{Q}$ as a local ring is an integral domain. Now suppose that K, K_i^* denote the respective quotient fields of the correspondingly taken completions of the rings Q, Q_i. The author identifies, applying the results of Chevalley and Zariski on local rings and the analytic irreducibility of normal varieties, the Kronecker product $E\times K$ over F with the full ring of quotients belonging to the completion of the semi-local ring Λ. Moreover $E\times K$ is semi-simple and the direct sum of the fields K_i^*, and thus a result for Dedekind rings in fields of kroneckerian dimension 1, viz. $\sum_{i=1}^h[K_i: K]=[E: F]$, is generalized considerably. Finally localization theorems for integral dependence are found. For example, integral dependence of elements of E with respect to Q is equivalent to simultaneous integral dependence with respect to the completions $\bar{Q}_i$. [See also O. Zariski, Theory and applications of holomorphic functions on algebraic varieties over arbitrary ground fields, Mem. Amer. Math. Soc., no. 5 (1951); these Rev. 12, 853.]

O. F. G. Schilling (Chicago, Ill.).

Barsotti, I. Local properties of algebraic correspondences. Trans. Amer. Math. Soc. 71, 349–378 (1951).

In classical algebraic geometry the theory of intersection multiplicities is usually based on the theory of algebraic systems, the intersection of two cycles z_1 and z_2 being defined by constructing two algebraic systems b_1 and b_2 containing z_1 and z_2 respectively as having the property that generic members of b_1 and b_2 intersect simply; $z_1\cap z_2$ is then defined (if it has the right dimension) by specialisation. It has to be proved that the definition of z_1 and z_2 is always possible, and that it does not depend on the choice of b_1 and b_2. Moreover, the theory usually has to assume that the carrier variety has no singular points. It is possible, however, to define intersections directly, and to use this definition as a basis for a theory of algebraic correspondences. This paper is concerned with this approach. A definition is given of the multiplicity of a geometrical ring (quotient ring of an irreducible sub-variety of an irreducible variety) valid for arbitrary base field, and on the basis of the definition a multiplicity can be assigned to any component of the intersection of two cycles, provided it is of the right dimension (even though the total intersection may be of higher dimension), and the properties of these intersections are used to develop a theory of algebraic correspondances (including the classical theory as a special case) in which it is possible to formulate results about the correspondent on a variety V of a point P on a variety F in a (non-degenerate) correspondance between F and G even when P is fundamental.

W. V. D. Hodge (Cambridge, England).

Please read the corresponding note on p. 1139.

Differential Geometry

***Favard, Jean. Élaboration des notions de courbe et de surface en géométrie différentielle.** Congrès International de Philosophie des Sciences, Paris, 1949, vol. III, Philosophie Mathématique, Mécanique, pp. 23–26. Actualités Sci. Ind., no. 1137. Hermann & Cie., Paris, 1951.

Fabricius-Bjerre, Fr. On the osculating conics of the cycloids. Mat. Tidsskr. B. 1951, 27–41 (1951).

This paper considers the cyclic curves, i.e., the hypocycloids, epicycloids, paracycloids, hypercycloids, involutes of circles and logarithmic spirals. Each of the curves is expressed by its polar-tangential equation $h=h(\theta)$, and it is shown that for all these curves the function $h(\theta)$ satisfies an equation of the form $h''+\lambda h=0$ where λ is a constant. Using this fact, the equation of the osculating conic at any given point P of the curve is determined, and it is found under what circumstances this conic is an ellipse, a parabola, or a hyperbola. For example, it is shown that for the hypocycloids and epicycloids, the osculating conics are all ellipses. The sextactic points, i.e., points where the osculating conic has at least fifth order contact, are determined. A similar treatment is given for curves which are polar reciprocals of the ones listed above with respect to a circle.

S. B. Jackson (College Park, Md.).

Myller, A. Les podaires négatives des courbes gauches. Acad. Repub. Pop. Române. Bul. Şti. A. 1, 29–36 (1948). (Romanian and French)

Viguier, Gabriel. Propriétés cycliques des développantes projectives. Revue Sci. 89, 183–185 (1951).

Terracini, A. La notion d'incidence de plans "infiniment voisins." Colloque de Géométrie Différentielle, Louvain, 1951, pp. 51–65. Georges Thone, Liège; Masson & Cie, Paris, 1951. 350 Belgian francs; 2450 French francs.

Se plaçant dans l'espace S_r d'ordre minimum ($r=5$) pour que l'incidence de deux plans ne soit pas une banalité (cas auquel on peut toujours se ramener par projection), l'auteur commence par envisager un système ∞^1 de plans et recherche les conditions pour qu'un plan du système soit incident en un point avec le plan consécutif. Il est ainsi conduit à introduire un entier σ qu'il appelle ordre d'approximation de l'incidence (conventionnellement infini si deux plans quelconques du système sont toujours incidents). σ ne peut admettre que les valeurs $2, 4, 6, \cdots, 14, 16, \infty$, et une étude précise de la structure des systèmes ∞^1 de plans pour les différentes valeurs possibles de σ montre que toutes les valeurs indiquées sont effectivement réalisables. L'auteur caractérise géométriquement les cas correspondants. Pour $\sigma\geqq2$ et $\sigma\geqq4$ la caractérisation ne présente aucune difficulté. Dans les autres cas elle met en jeu des hypersurfaces devant contenir un certain nombre de plans consécutifs, hypersurfaces qui sont des hyperquadriques jusqu'à $\sigma\geqq8$ et des hypercubiques à partir de $\sigma\geqq10$. La notion d'ordre d'approximation de l'incidence peut être étendue aux systèmes ∞^α de dimension $\alpha>1$. On a alors généralement $\alpha=2$, et l'étude de la possibilité $\alpha\geqq4$ met en jeu la surface de Véronèse et des congruences W hyperspatiales à nappes focales applicables projectivement au 3ème ordre.

Pour les systèmes ∞^1, la considération de quadriques non singulières jouant le rôle de la quadrique de Klein, conduit à des résultats intéressant d'autres théories, et notamment celle des surfaces réglées de l'espace ordinaire.

P. Vincensini (Marseille).

Pylarinos, O. Über die Strahlensysteme, deren Brennflächenmäntel durch die Systemstrahlen konform aufeinander abgebildet werden. Arch. Math. **2** (1949–1950), 449–455 (1951).

Le référent a étudié sous le nom de congruences (ω), les congruences rectilignes dont les plans focaux font l'angle constant ω [Vincensini, Acta Math. **71**, 145–174 (1939); ces Rev. **1**, 173]. L'article actuel étudie la correspondance ponctuelle établie par les couples de foyers associés sur les deux nappes d'une congruence (ω), en vue de rechercher les cas où cette correspondance est conforme. L'auteur est conduit aux résultats suivants. La représentation des deux nappes l'une sur l'autre ne peut être conforme que si $\omega=2\pi/3$, et les congruences jouissant de la propriété indiquée se subdivisent en deux classes. Pour les congruences de l'une des classes les images sphériques des développables forment deux faisceaux de loxodromies coupant les méridiens respectivement sous les angles $-\pi/3$ et $+\pi/3$. Les surfaces focales des congruences de la deuxième classe sont des surfaces W. Si en particulier la correspondance entre les deux nappes est une isométrie, les deux nappes sont des surfaces W dont les rayons de courbure principaux vérifient la même relation. *P. Vincensini* (Marseille).

Geidel'man, R. M. On some properties of ruled images of congruences of circles. Uspehi Matem. Nauk (N.S.) **6**, no. 4(44), 162–169 (1951). (Russian)

An interesting, partly expository, paper dealing with various types of congruences of circles in Euclidean three-space. The author uses pentaspherical coordinates and takes advantage of the well known fact that the conformal group of three-space is simply isomorphic to a subgroup of the projective group of P_4 leaving invariant a hyperquadric Q_3. In the correspondence of Darboux, the pentaspherical coordinates of a sphere are the homogeneous coordinates of a point in P_4 so that a pencil of spheres having a circumference in common corresponds to a straight line. The paper is mainly concerned with congruences of circles which correspond to various line complexes in P_4. Thus a rectilinear congruence in P_4 is the correspondent of a cyclic system of Ribaucour if the foci of every line are conjugate with respect to the fixed quadric Q_3. *M. S. Knebelman.*

Geidel'man, R. M. Conformal bending of a three-dimensional complex of circles. Doklady Akad. Nauk SSSR (N.S.) **80**, 149–152 (1951). (Russian)

In conformal 3-space let K be a complex of circles depending on three parameters. Two such complexes K and $\mathring{K}$ are conformally applicable to order n if there exists between their circles a one-to-one correspondence such that with each pair of corresponding circles $[S_2S_3]$ and $[\mathring{S}_2\mathring{S}_3]$ is associated a conformal transformation T which carries $\mathring{K}$ into a complex $\bar{K}$ so that the circle $[S_2S_3]$ coincides with $[\bar{S}_2\bar{S}_3]$ and all neighboring circles coincide with their corresponding ones up to and including magnitudes of order n. A circle complex that admits such applicability is deformable and it is known that any complex is not deformable of order 2 and that a general complex is not deformable. By considering a canonical repère of Cartan—consisting of two points and three mutually orthogonal unit spheres, S_1, S_2, S_3, S_2 and S_3 containing the circle—the author investigates applicability of order 1 and proves that applicability of order 1 depends on 6 arbitrary functions of one parameter. Since spheres in a conformal M_3 correspond to points in projective P_4, the above result applies to 3-parameter line complexes in P_4. *M. S. Knebelman* (Pullman, Wash.).

Tomonaga, Yasuro. Connectionization of Laguerre geometry. Sūgaku (Mathematics) **2**, 297–311 (1950). (Japanese)

The author develops here the theory of connection of a space whose tangent spaces are Laguerre spaces. In an n-dimensional Euclidean space, the equation of a sphere with center V^i $(i, j, k, \cdots=1, 2, \cdots, n)$ and with radius V^0 is given by $g_{ij}(X^i-V^i)(X^j-V^j)-(V^0)^2=0$ or

$$g_{\lambda\mu}(X^\lambda-V^\lambda)(X^\mu-V^\mu)=0 \quad (\lambda, \mu, \nu, \cdots=0, 1, \cdots, n),$$

where $g_{i0}=g_{0i}=0$, $g_{00}=-1$, $X^0=0$, and so the length D of the common tangent to two spheres U^λ and V^λ is given by

$$(1) \qquad D^2=g_{\lambda\mu}(U^\lambda-V^\lambda)(U^\mu-V^\mu).$$

Moreover, a necessary and sufficient condition that a hyperplane $t_iX^i-p=0$ $(g^{ij}t_it_j=1)$ be tangent to a sphere V^λ is that $t_\lambda X^\lambda=p$ $(t_0=-1)$. Laguerre geometry is defined as the theory of invariants under a linear group on V^λ which does not change (1). A transformation of this group carries a hyperplane into a hyperplane and a sphere into a sphere, keeps invariant the contact relation between a hyperplane and a sphere and does not change the length of the common tangent to two spheres.

Now, we take an n-dimensional Riemannian space with metric $ds^2=g_{jk}dx^jdx^k$ and consider the so-called tangent spaces as Laguerre spaces in the sense explained above. Next, we define the Laguerre connection by the condition that, a field of spheres $V^\lambda(x)$ being given, the image of the sphere $V^\lambda+dV^\lambda$ at (x^i+dx^i) in the tangent space at (x^i) is given by $V^\lambda+\delta V^\lambda+dx^\lambda$, where $\delta V^\lambda=dV^\lambda+\Gamma^\lambda{}_{\mu k}V^\mu dx^k$ and $dx^0=0$, and the condition that $\delta g_{\lambda\mu}=0$. If we assume moreover that the space has no torsion, then the $\Gamma^i{}_{jk}$ become Christoffel symbols and we have $\Gamma^0{}_{jk}=\Gamma^0{}_{kj}$, $\Gamma^i{}_{0k}=g^{ij}\Gamma^0{}_{jk}$, $\Gamma^0{}_{0k}=0$.

The author then discusses the theory of one-parameter families of spheres (Frenet-Serret formulas), the theory of $(n-1)$-parameter families of spheres (formulas in hypersurface theory), holonomy groups, some problems arising from the variation of Laguerre length, and the theory of infinitesimal deformations.

In the last paragraph, the author treats the case in which $\Gamma^0{}_{jk}=-\Gamma^0{}_{kj}$ and shows that in this case the theory is equivalent to the theory of the spaces developed by Einstein and Mayer [S.-B. Preuss. Akad. Wiss. Phys.-Math. Kl. **25**, 541–557 (1931)] to establish a unified field theory.

K. Yano (Princeton, N. J.).

✓★**Cartan, Élie. Leçons sur la géométrie des espaces de Riemann.** 2d ed. Gauthier-Villars, Paris, 1951. viii+378 pp. 1800 frs.

Nouveau tirage of a book first issued in 1928.

Kanitani, Jōyō. Sur la connexion affine admettant d'une métrique. Mem. Coll. Sci. Univ. Kyoto Ser. A. Math. **26**, 95–112 (1951).

Given a space R_n generated by a point x^i $(i=1, \cdots, n)$ with an affine connection $\Gamma^h{}_{ij}d\omega^j$, where $d\omega^i$ $(i=1, \cdots, n)$ are distinct homogeneous linear expressions of $dx^1, \cdots, dx^n$ and $\Gamma^h{}_{ij}$ are analytic functions of $x^1, \cdots, x^n$. This paper is devoted to a study of conditions for the existence of a quadratic differential form $ds^2=g_{ij}d\omega^id\omega^j$, such that $\Gamma^h{}_{ij}$ can be expressed by

$$\Gamma^h{}_{ij}=\tfrac{1}{2}g^{hl}\left(\frac{\partial g_{jl}}{\partial\omega^i}+\frac{\partial g_{il}}{\partial\omega^j}-\frac{\partial g_{ij}}{\partial\omega^l}-g_{im}a^m{}_{jl}-g_{jm}a^m{}_{il}-g_{lm}a^m{}_{ij}\right),$$

where $a^h{}_{ij}$ are the coefficients of a covariant bilinear expression of $d\omega^h$, namely, $(\delta, d)\omega^h = a^h{}_{ij}d\omega^i\delta\omega^j$. The discussion is divided into three different cases according as $n=2$, $n=3$ or $n>3$, and the required conditions are derived and studied in detail for each case. *C. C. Hsiung.*

Verbickiĭ, L. L. A tensor criterion of conformally Euclidean spaces of class 1. Doklady Akad. Nauk SSSR (N.S.) 81, 133–136 (1951). (Russian)

A V_n with a positive definite form $ds^2=g_{ij}dx^idx^j$ will be conformally euclidean and of class 1 ($n\geqq 4$) if its second fundamental form is $\pi_{ij}=\sigma g_{ij}+(\bar\sigma-\sigma)e_ie_j$, where e_i is a unit vector in V_n and σ and $\bar\sigma$ are the curvatures of V_n. In this case

$$R_{ijkl}=\sigma^2(g_{ik}g_{jl}-g_{il}g_{jk})+\sigma(\bar\sigma-\sigma)(g_{ik}e_je_l+g_{jl}e_ie_k-g_{il}e_ke_j-g_{jk}e_ie_l).$$

M. S. Knebelman (Pullman, Wash.).

Verbickiĭ, L. L. Construction of a conformally Euclidean space of class 1 within a containing Euclidean space. Doklady Akad. Nauk SSSR (N.S.) 81, 333–336 (1951). (Russian)

The paper reviewed above dealt with tensorial characterization of conformally Euclidean V_n ($n\geqq 4$) of class 1. This note gives a canonical form for the metric of such a space. The main theorem established in this note is that a conformally Euclidean V_n of class 1 may be defined by $ds^2=[\psi^2(dy^1)^2+\sum_2^n(dy^i)^2]/U^2$ where U and ψ are quadratic polynomials in $y^2, y^3, \cdots, y^n$ with coefficients that are functions of y^1:

$$U=\tfrac{1}{2}a\sum_2^n(y^i)^2+\sum_2^n b_iy^i+c \quad\text{and}\quad \psi=\tfrac{1}{2}A\sum_2^n(y^i)^2+\sum_2^n B_iy^i+C.$$

If a, b_i, c are not all constants then $\psi=\phi\partial U/\partial y^1$ where $\phi=\phi(y^1)$ with $\sigma^2(y^1)=2ac-\sum_2^n b_i^2-\phi^{-2}>0$, $\sigma\neq$const. If a, b_i, c are all constants then $\sigma^2=2ac-\sum_2^n b_i^2>0$ and $\sum_2^n B_ib_i-aC-Ac\neq 0$. *M. S. Knebelman.*

Samelson, Hans. A theorem on differentiable manifolds. Portugaliae Math. 10, 129–133 (1951).

The following theorem is proved: If a compact differentiable manifold admits a continuous field of r-dimensional contact elements (i.e., r-dimensional subspace of the tangent space for odd r), then its Euler-Poincaré characteristic is zero. *S. Chern* (Chicago, Ill.).

Chern, Shiing-shen, and Spanier, E. A theorem on orientable surfaces in four-dimensional space. Comment. Math. Helv. 25, 205–209 (1951).

For a closed oriented differentiable surface M in Euclidean E^4 one constructs maps t and n of M into the Grassmann manifold $G(2, 2)$ by assigning to each point p of M the plane through the origin, parallel to the tangent, resp. normal, plane to M at p. The manifold $G(2, 2)$ is homeomorphic with the product $S_1\times S_2$ of two 2-spheres. For the basic cycle of M one has therefore $t_*(M)=u_1S_1+u_2S_2$ with integers u_1, u_2. Blaschke [Ann. Mat. Pura Appl. (4) 28, 205–209 (1949); these Rev. 11, 741] showed that u_1+u_2 is the characteristic $\chi(M)$, and stated $u_1=u_2$. The authors give a new proof of the first statement, and prove the second. If Ω is the characteristic cocycle of the Stiefel manifold $V_{4,2}$ over $G(2, 2)$, then $t^*\Omega$ and $n^*\Omega$ are the characteristic cocycles of the tangent and normal bundle of M. Now Ω can be determined explicitly, $t^*\Omega$ is essentially $\chi(M)$, and $n^*\Omega$ is 0; the theorem follows easily from these facts. For the equation $n^*\Omega=0$, and for the more general proposition that the characteristic cocycle of the normal bundle of a manifold M in a manifold M' is zero, if M is ~ 0 in M', a simple proof, due to H. Hopf, is given. *H. Samelson* (Ann Arbor, Mich.).

Myers, S. B. Curvature of closed hypersurfaces and nonexistence of closed minimal hypersurfaces. Trans. Amer. Math. Soc. 71, 211–217 (1951).

Im 3-dimensionalen euklidischen Raum gelten bekanntlich die folgenden Sätze: Eine geschlossene (mehrmals differenzierbare) Fläche besitzt Punkte mit positiver Krümmung; die Krümmung einer Minimalfläche ist nirgends positiv; es gibt keine geschlossene Minimalfläche. In der vorliegenden Arbeit handelt es sich um die Übertragung dieser Sätze auf Hyperflächen in einer Riemannschen n-dimensionalen Mannigfaltigkeit R_n. Ist R_{n-1} eine Hyperfläche in R_n, so wird unter der Relativkrümmung eines in R_{n-1} gelegenen 2-dimensionalen Flächenelementes F der Überschuss der Riemannschen Krümmung von R_{n-1} über diejenige von R_n in F verstanden. Die gewünschte Übertragung der obigen Sätze wird durch den Umstand erschwert, dass für einen Punkt $O\epsilon R_n$ die folgende Menge $M(O)$ im allgemeinen nicht leer ist: der Ort derjenigen Punkte, in denen die von O ausgehenden geodätischen Linien aufhören, die absolut kürzesten Verbindungen mit O zu sein.

Die Hauptsätze, die erhalten werden, lauten: Jeder Punkt $O\epsilon R_n$ hat eine solche Umgebung $N(O)$, dass jede geschlossene $R_{n-1}\subset N(O)$ Punkte besitzt, in denen alle Relativkrümmungen positiv sind. Der Raum R_n sei vollständig und seine Riemannsche Krümmung nirgends positiv; O sei ein fester Punkt in R_n und R_{n-1} eine geschlossene Hyperfläche; dann sind in denjenigen Punkten von R_{n-1}, in welchen die Entfernung von O ein Maximum erreicht, welche aber nicht auf $M(O)$ liegen, alle Relativkrümmungen von R_{n-1} positiv. Das Analoge gilt, wenn die Krümmung von R_n eine positive Konstante K und wenn alle Punkte von R_{n-1} von dem Punkt O eine Entfernung $<\pi/2K^{\frac{1}{2}}$ haben. Die Ricci-Krümmung einer Minimalmannigfaltigkeit $R_m\subset R_n$ ist nirgends grösser als die Ricci-Krümmung ihrer m-dimensionalen geodätischen Tangentialmannigfaltigkeit. ("Minimalmannigfaltigkeit": Extremale in Bezug auf das m-dimensionale Volumen.) Ist R_n vollständig und von nirgends positiver Krümmung, R_{n-1} eine geschlossene Minimalmannigfaltigkeit in R_n, so liegen alle Punkte von R_{n-1}, in denen die Entfernung von einem festen Punkt $O\epsilon R_n$ ein Maximum erreicht, auf $M(O)$. Das Analoge gilt, wenn die Krümmung von R_n eine positive Konstante K und R_{n-1} überall weniger als $\pi/2K^{\frac{1}{2}}$ von O entfernt ist. Die Sätze sind in naheliegender Weise zu vereinfachen, wenn die Mengen $M(O)$ leer sind; dies ist insbesondere der Fall, wenn R_n vollständig, von nirgends positiver Krümmung und einfach zusammenhängend ist; z.B. folgt unmittelbar, dass in einem solchen Raum keine geschlossenen Minimalmannigfaltigkeiten R_{n-1} existieren. *H. Hopf* (Zürich).

✓***Lichnerowicz, A. Courbure, nombres de Betti, et espaces symétriques.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 216–223. Amer. Math. Soc., Providence, R. I., 1952.

The author summarizes results from the topic "curvature and Betti numbers", but only as referring to real compact spaces V_n with a positive-definite Riemannian metric. Of the author's own contributions we might single out the following ones (assuming also V_n orientable, which is not

always needed). (1) If in addition to the ground tensor $g_{\alpha\beta}$ there is another tensor $T_{\alpha\beta}$ not proportional to it, but $T_{\alpha\beta,\gamma}=0$, then the Betti-number is $\geqq k_p$, where k_p is the number of eigenvalues of the matrix $T_{\alpha\beta}$ of multiplicity p, this number being independent of the point in space. (2) On V_4, if the Ricci curvature vanishes, $R_{\alpha\beta}=0$, and the Euler-Poincaré characteristic is <2, then the space is flat (it is not known, even for a compact Kähler space V_{2k} whether $R_{\alpha\bar\beta}=0$ implies flatness always). (3) If V_n is locally imbeddable isometrically into Euclidean E_{n+1}, and the so-called second fundamental form is positive-definite, then the first and second Betti numbers are 0. (4) Form the four-indices-tensor

$$H_{\alpha\beta\gamma\delta\lambda\mu}=R_{\alpha\beta\gamma\delta,\lambda,\mu}-R_{\alpha\beta\gamma\delta,\mu,\lambda}$$

and then the scalar

$$K=g^{\alpha\lambda}H_{\alpha\beta\gamma\delta\lambda\mu}R^{\mu\beta\gamma\delta}.$$

Then for $K\geqq 0$, if also $R_{\alpha\beta,\gamma}=0$, V_n is symmetric in the sense of E. Cartan; and, if $R_{\alpha\beta}=0$, then the space is flat. (5) If there exists a vector field R_λ such that

$$R_{\alpha\beta\gamma\delta,\lambda}=k_\lambda R_{\alpha\beta\gamma\delta}$$

(spaces of Ruse and A. G. Walker), in which case k_λ is the gradient of a scalar field log P, then either $P^2=R^2$ or else the space is symmetric. If $R_{\alpha\beta}=0$, then the space is flat.

S. Bochner (Princeton, N. J.).

Milgram, A. N., and Rosenbloom, P. C. Heat conduction on Riemannian manifolds. II. Heat distribution on complexes and approximation theory. Proc. Nat. Acad. Sci. U. S. A. **37**, 435–438 (1951).

Continuing part I [same Proc. **37**, 180–184 (1951); these Rev. **13**, 160], the authors further discuss the smoothing of cocycles alias differential forms by making them solutions of the heat equations $\Delta\alpha=\partial\alpha/\partial t$ in $0<t<\infty$, with given "boundary values" for $t\rightarrow 0$, and also the "mapping" of a chain into a time-dependent form. The leading result is that by taking the opposite limit $t\rightarrow\infty$, any chain C^p gives rise to a form $HC^p=\lim_{t\to\infty} T_tC^p$ which is always harmonic. If $Z_1^p, \cdots, Z_\beta^p$ is a basis of p-cycles then $HZ_1^p, \cdots, HZ_\beta^p$ are linearly independent harmonic p-forms. *S. Bochner*.

Kawaguchi, Akitsugu. Theory of areal spaces. Sūgaku (Mathematics) **3**, 76–81 (1951). (Japanese)

The author calls an n-dimensional space in which the area on an m-dimensional subspace $x^i=x^i(u^1, \cdots, u^m)$ is given by the integral

$$S=\int_{(m)} F\left(x^i, \frac{\partial x^i}{\partial u^\alpha}\right)du^1du^2\cdots du^m$$

an areal space. This contains, of course, the Finsler and the Cartan spaces as special cases. The paper is expository. In §1, the author reviews the origin and subsequent development of the theory of areal spaces. In §2, he discusses the determination of the fundamental metric tensor, the problem which is solved for some special cases, but not yet solved completely in the general case. In §3, he describes two special cases, the areal space of metric class and that of submetric class. The former was studied by R. Debever [C. R. Acad. Sci. Paris **224**, 887–889, 1269–1271 (1947); these Rev. **8**, 491] and the latter by the author (not yet published). In §4, the author discusses the problem of the determination of the connection in a general areal space. In the last §, he states some important problems to be solved in the theory of areal spaces. *K. Yano* (Princeton, N. J.).

De Sloovere, Henri. Sur le nombre d'invariants distincts, fonctions de tenseurs, d'après la méthode de Lie et De Donder. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **37**, 583–598 (1951).

The author's purpose is to obtain a complete system of partial differential equations for the determination of invariants I which depend upon: (1) a symmetric third order tensor; (2) a vector and its first and second derivatives. To solve these problems, the author introduces the Lie infinitesimal transformation which generates the group of transformations and a partial derivative operator used by De Donder [Théorie invariantive du calcul des variations, Gauthier-Villars, Paris, 1935]. With the aid of these tools, the author determines operators whose commutators are linearly expressible in terms of the operators and, thus, these operators form the desired complete system. In the first example, the desired operators are easily determined. However, in the second example, extensive calculations are required in order to show that the chosen operators satisfy the commutator conditions. *N. Coburn*.

Nalli, Pia. Trasporti rigidi di vettori negli spazi quadridimensionali. Ann. Mat. Pura Appl. (4) **28**, 291–297 (1949).

The author continues the discussion of the conditions to be satisfied by a skew-symmetric tensor A_{ij} in order that it may be of the form $v_iv_j'-v_jv_i'$, where v_i is a versor defined along a curve of parameter t and $v_i'=dv_i/dt$ [Ann. Mat. Pura Appl. (4) **26**, 67–72 (1947); these Rev. **10**, 149]. Three conditions are obtained involving only the first and second derivatives of A_{ij}. *A. J. McConnell* (Dublin).

✓*Schouten, J. A. **Tensor Analysis for Physicists.** Oxford, At the Clarendon Press, 1951. x+275 pp. $6.00.

This text contains an excellent treatment of tensor analysis and applications to linear elasticity theory, dynamics, relativity, and quantum mechanics. Though the text does not assume a prior acquaintanceship with tensor theory, it is desirable that the reader possess some background in the subject. For such a reader as well as for the specialist, this text will furnish a wealth of useful and unified information.

The subject matter is divided into two parts: (1) the development of tensor analysis in E_n (an n-dimensional space determined by the general affine group of coordinate transformations, G_a) and in X_n (an n-dimensional space determined by the general group of coordinate transformations of class C^2 and non-vanishing Jacobian); (2) the application of this analysis to the previously indicated subjects (elasticity, etc.). In the first part, the stress is placed upon those developments which have a bearing on the applications.

First, the author studies the geometry of E_n. In this section, the notation (kernel-index method) and the role of the group G_a and its subgroups (the equivolumar, orthogonal, etc.) in determining a geometry are explained. The author's diagrammatic representations of covariant vectors, contravariant vectors, and of bivectors and trivectors in E_3 and the identifications of these representations for subgroups of G_a are both interesting and valuable. After a study of tensor algebra, and the normal forms of symmetric and alternating second order tensors, the metric tensor is introduced (thus, E_n becomes a Euclidean space, R_n) and the raising and lowering of indices and also orthogonal components are discussed. The local geometry of X_n is that of E_n. To construct the derivative of geometric quantities (tensors and tensor

densities) in X_n, which does not possess a connection, the author deals with p-vectors. With the aid of these alternating tensors, a general formulation of Stoke's theorem is obtained. This is followed by a study of the Lie and Lagrange derivatives and anholonomic (often called non-holonomic) coordinates in X_n. After introducing a connection or linear displacement in X_n (which becomes an L_n) the author discusses geodesic and normal coordinates. For the V_n (L_n with a fundamental tensor) it is shown that the first variation of arc length vanishes along a geodesic. Further, by use of parallel transport along a circuit, the author introduces the curvature tensor and then examines the various identities satisfied by this tensor.

Before discussing the applications of tensor analysis, the author considers physical objects and their dimensions. The theory is based upon the transformation relation between the contravariant (or covariant) components and the orthogonal components (which are the physical components when the n-tuple is appropriately chosen) of a geometric object. The dimension of the contravariant components furnishes the absolute dimensions; the dimension of the orthogonal components is said to determine the relative (or physical) dimension. By use of this scheme, the author considers the absolute and relative dimensions of various physical quantities.

The principal problems treated in the elasticity section are the determination of the elastic constants for the various crystal classes and the study of waves in a homogeneous anisotropic medium. After introducing the displacement vector and the strain tensor, the author considers the vector density associated with a two-dimensional facet and then the stress tensor density. In keeping with this approach, the mass density (or mass per measuring parallelopiped) is introduced. An alternative approach is to work with the stress tensor and the scalar mass (or mass per unit volume). By examining the energy relation, it is shown that the elastic coefficients are the components of a tensor-tensor density (a fourth order tensor density which is symmetric in each index of a pair of indices as well as in the pairs). In addition, the dialectric and piezo-electric constants are shown to be the components of tensors. The study of the structure of the elastic, dialectric and piezo-electric tensors for the various crystal classes depends upon the theory that these tensors are invariant under the group of transformations which leaves the crystal invariant. The results are furnished in the form of tables. In the study of waves, the author represents the displacement vector in terms of its amplitude and phase. By use of physical approximations, the two relations between the displacement vector and the normal to the wave front are obtained. Conditions for self-reciprocity of various crystals are examined. The chapter concludes with a study of the quartz resonator.

The theory of classical dynamics of a system of particles is developed for: (1) holonomic systems which are either scleronomic (Cartesian coordinates independent of time) or rheonomic (Cartesian coordinates time dependent); (2) nonholonomic systems. The rheonomic holonomic systems are discussed by introducing an appropriate connection in an affine film space A_{n+1}. Non-holonomic (or anholonomic) systems are treated by introducing the anholonomic object. Further, the equations of Lagrange, Hamilton and the integration of the Hamilton-Jacobi equation are investigated.

In his treatment of special relativity theory, the author's approach is to start with the classical equations and then determine the modifications of these equations which are necessary in order that they remain invariant under the Lorentz transformation. The electrodynamic force equations, the Newtonian equations of motion, and the hydrodynamical equations of motion are treated in this manner. The extensive symbolism leads to elegance of presentation of the various ideas but requires careful reading.

The last chapter deals with those elements of unitary geometry which are needed for an introduction to Dirac's matrix calculus. By introducing complex linear coordinate transformations and a hermitian tensor of rank n, the E_n becomes a U_n. Anticipating terms used by Dirac, the author introduces two types of vectors in U_n, the "ket" and the "bra". The reason for these names and the relation of these vectors to the generalized vectors of these types in the Dirac calculus is explained. The actual transition from U_n to the infinite-dimensional space used by Dirac is made by use of matrices. *N. Coburn* (Ann Arbor, Mich.).

NUMERICAL AND GRAPHICAL METHODS

✱**Tables of the error function and of its first twenty derivatives.** By the Staff of the Computation Laboratory. The Annals of the Computation Laboratory of Harvard University, vol. 23. Harvard University Press, Cambridge, Mass., 1952. xxviii+276 pp. $8.00.

Let

$$\phi^{(-1)}(x)=(2\pi)^{-1/2}\int_0^x e^{-t^2/2}dt, \quad \phi^{(n)}(x)=\frac{d^n}{dx^n}[(2\pi)^{-1/2}e^{-x^2/2}],$$
$$n=0, 1, 2, \cdots.$$

The functions $\phi^{(n)}(x)$ are tabulated as follows.

Table I:	$n=-1, \cdots, 4$ for $x=0.000(0.004)6.468$; 6D.
Table II:	$n=5, \cdots, 10$ for $x=0.000(0.004)8.236$; 6D.
Table III:	$n=11, \cdots, 15$ for $x=0.000(0.002)9.610$; 7S.
Table IV:	$n=16, \cdots, 20$ for $x=0.000(0.002)10.902$; 7S.

In addition, there is a short discussion of the functions $\phi^{(n)}(x)$, of the computation of the tables, and of the uses of the functions. The zeros of $\phi^{(n)}(x)$, $n=1, \cdots, 20$, are given to 10D. *J. V. Wehausen* (Providence, R. I.).

✓**Nath, Pran. Confluent hypergeometric function.** Sankhyā 11, 153–166 (1951).

Tables to seven significant places of

$$M(\alpha, \gamma, x)=1+\frac{\alpha}{1\cdot\gamma}x+\frac{\alpha(\alpha+1)}{1\cdot 2\cdot\gamma(\gamma+1)}x^2+\cdots$$

for $\gamma=3$, $\alpha=1(1)40$, $x=.02(.02).1(.1)1(1)10(10)50, 100, 200$ and $\gamma=4$, $\alpha=1(1)50$, and the same values of x as in the previous case. *A. Erdélyi* (Pasadena, Calif.).

✱**Spencer, Roy C., and Reynolds, George E. A table of normalized parabolic coordinates and arc lengths.** Air Force Cambridge Research Center, Cambridge, Mass., 1951. 10 pp.

This table is intended to facilitate the design of parabolic reflectors and gives to 5 decimals the y coordinate and arc length s from the origin of a point (x, y) on the parabola $y=\frac{1}{4}x^2$ for $x=0(.01)2$. There are two graphs of these functions giving s as a function of x and s, x as functions of y. *D. H. Lehmer* (Los Angeles, Calif.).

Laible, Theodor. Höhenkarte des Fehler-integrals. Z. Angew. Math. Physik 2, 484–486 (1951).

Two altitude charts of the function

$$\operatorname{erf}(z) = 2\pi^{-\frac{1}{2}} \int_0^z \exp(-t^2)dt$$

of the complex variable $z = x + iy$ are given. The one with more detail is for $0 < x < 1.8$, $0 < y < 2.3$. A table of the first 6 zeros of erf (z) is given to 3 decimal places.

D. H. Lehmer (Los Angeles, Calif.).

Blanch, G., and Yowell, E. C. A guide to tables on punched cards. Math. Tables and Other Aids to Computation 5, 185–212 (1951).

✓*Kahn, Herman. Modification of the Monte Carlo method.** Proceedings, Seminar on Scientific Computation, November, 1949, pp. 20–27. International Business Machines Corp., New York, N. Y., 1950.

The statistical sampling (Monte Carlo) procedures for the numerical solution of mathematical problems are based on an interpretation of the quantity to be calculated as the expected value of some random variable. The random variable and its probability distribution are not uniquely determined by the mathematical problem. The author discusses, for the problem of evaluating multidimensional integrals and for the solution of Fredholm integral equations, methods of choosing the random process so as to make the standard error as small as is practically possible. This amounts to finding a method of sampling that scans most carefully the regions that contribute most to the error. Hence the name "importance sampling" for this procedure. It is stated, without proof, that finding the theoretically best method of importance sampling for an integral equation is equivalent to solving the adjoint problem. For the actual computation the decision has, however, to be made on the basis of physical intuition and preliminary approximations, for which some suggestions are given. *W. R. Wasow.*

✓*Ulam, S. On the Monte Carlo method.** Proceedings of a Second Symposium on Large-Scale Digital Calculating Machinery, 1949, pp. 207–212. Harvard University Press, Cambridge, Mass., 1951. $8.00.

This is a general description of the Monte Carlo method, which is the use of sampling procedures to estimate such quantities as complicated probabilities, definite integrals over the unit cube in n dimensions, and solutions of partial differential equations of diffusion type. The author stresses the need for "importance sampling" in all these applications. He illustrates the possibility of expressing solutions of other types of partial differential equations in terms of properties of solutions of equations of diffusion type, so that more equations may respond to a Monte Carlo approach. It is pointed out that a formal system of mathematics can be described geometrically, and that a theorem then asserts that a certain set is vacuous. It is suggested that a difficult theorem might be approached heuristically by the attempt to construct points of the set by certain random choices in n dimensions (Monte Carlo). Failure would lend credibility to the theorem. *G. E. Forsythe* (Los Angeles, Calif.).

***Pérès, Joseph. Méthode et calcul analogique.** Congrès International de Philosophie des Sciences, Paris, 1949, vol. III, Philosophie Mathématique, Mécanique, pp. 109–120. Actualités Sci. Ind., no. 1137. Hermann & Cie., Paris, 1951.

✓*Lehmer, D. H. Mathematical methods in large-scale computing units.** Proceedings of a Second Symposium on Large-Scale Digital Calculating Machinery, 1949, pp. 141–146. Harvard University Press, Cambridge, Mass., 1951. $8.00.

The aim of this paper is to discuss in a general way certain features of the mathematics that are characteristic of the large scale digital computing machines. Special attention is given to the problem of generating random numbers. The author's proposal for generating a sequence of "random" 8 decimal numbers u_n is to start with some number $u_0 \neq 0$, multiply it by 23, and subtract the two digit overflow on the left from the two right hand digits to obtain a new number u_1. By repeating this process he generates a sequence u_n which he shows has a repetition period of 5,882,352. An IBM 602A was used to generate 5000 numbers of such a sequence. Four standard tests (unspecified) were applied to check the "randomness" and the sequence passed all four. The author observes, however, that all of the members of the sequence he chose were divisible by 17.

R. Hamming (Murray Hill, N. J.).

***Hastings, Cecil, Jr. Rational approximation in high-speed computing.** Proceedings, Computation Seminar, December 1949, pp. 57–61. International Business Machines Corp., New York, N. Y., 1951.

For a given transcendental function the author finds a family of rational approximations. These approximations are the best possible in the sense that the maximum error is kept a minimum for the form of the approximation used. The approximations have the additional property that they are "computable" in the sense that at the various stages in the evaluation the numbers calculated stay within a reasonable range of magnitudes. In this paper a number of these approximations are illustrated and some of the theory behind the methods is worked out. *R. Hamming.*

✓*Bramble, C. Clinton. Empirical study of effects of rounding errors.** Proceedings of a Second Symposium on Large-Scale Digital Calculating Machinery, 1949, pp. 147–151. Harvard University Press, Cambridge, Mass., 1951. $8.00.

The author first discusses and tabulates the probability distribution of the units and tens digits in the product of two independent integers whose units and tens digits are uniformly distributed. The uniformizing effect of "carry" is pointed out. For the ambiguous case in Gaussian round-off it is proposed that integers ending in 5 be rounded off to the nearest odd multiple of 10. This has the advantage of tending to uniformize the distribution of the tens digits of products of integers; for example, it exactly equalizes the numbers of odd and even digits. The author next sketches the accumulated errors resulting from small errors in initial data or in the recursion formulas of certain actual numerical integrations (these are matters of "stability" more than of "round-off"). It is stated that, in the inversion of certain matrices (not described) of orders 6 to 10 on the Aiken Relay Calculator, 2 to 4 significant figures were lost by round-off.

G. E. Forsythe (Los Angeles, Calif.).

Salzer, H. E. Formulas for finding the argument for which a function has a given derivative. Math. Tables and Other Aids to Computation 5, 213–215 (1951).

★**Frenkiel, F. N., and Polachek, H. An algorithm for fitting a polynomial through n given points.** Proceedings, Computation Seminar, December 1949, pp. 71–73. International Business Machines Corp., New York, N. Y., 1951.

Formulas are given for the computation of the coefficients of the polynomial $a_1+a_2x+a_3x^2+\cdots+a_nx^{n-1}$ which passes through n arbitrary points. *E. Frank* (Chicago, Ill.).

★**Alt, Franz L. Machine methods for finding characteristic roots of a matrix.** Proceedings, Computation Seminar, December 1949, pp. 49–53. International Business Machines Corp., New York, N. Y., 1951.

"The purpose of this paper is to describe a few expedients which can be applied to computation of characteristic roots of matrices by means of punched card machines. In the course of two problems of this kind, recently handled by the Computation Laboratory of the National Bureau of Standards, some of these methods or variants of methods were actually tried out on cards, and some others were considered and laid out without actually being carried through." *R. Hamming* (Murray Hill, N. J.).

★**Murray, Francis J. Simultaneous linear equations.** Proceedings, Scientific Computation Forum, 1948, pp. 105–106. International Business Machines Corp., New York, N. Y., 1950.

This paper contains a brief discussion of the solution of "ill-conditioned equations" as practiced at the Watson Laboratory in conjunction with a simultaneous equation solving machine. *R. Hamming* (Murray Hill, N. J.).

★**Kunz, Kaiser S. Matrix methods.** Proceedings, Computation Seminar, December 1949, pp. 37–42. International Business Machines Corp., New York, N. Y., 1951.

A clear review of four known methods for solving linear equations: the elimination method; its variant, the square root method; Jacobi's method of "ordinary iteration" (the "Gesamtschrittverfahren"); and Seidel's method (the "Einzelschrittverfahren"). There are remarks on finding characteristic values of matrices. (The author says Jacobi's method is sometimes called the "Gauss-Seidel method," but the reviewer has seen the latter name used only for Seidel's method. The author says Seidel's method is due to Morris!) *G. E. Forsythe* (Los Angeles, Calif.).

Chow, Carl. Gradual developing method. Bull. Géodésique 1951, 221–229 (1951).

The author describes a method for solving linear equations which may be viewed as a generalization of the familiar Gauss elimination method. Instead of solving for one variable at a time he solves for a small group and eliminates these. The paper may be viewed as preliminary in the sense that the author has not yet made a stability analysis of the method. *H. H. Goldstine* (Princeton, N. J.).

Crandall, Stephen H. On a relaxation method for eigenvalue problems. J. Math. Physics 30, 140–145 (1951).

Die Arbeit befasst sich mit einem Verfahren zur Eingrenzung von Eigenwerten bei Matrizen. Es gründet sich auf einen Einschliessungssatz, der in dem vor kurzem erschienenen Buch von L. Collatz [Eigenwertaufgaben mit technischen Anwendunge, Ankademische Verlagsgesellschaft, Leipzig, 1949, Seite 289; diese Rev. 11, 137] angegeben wird. Zu Beginn der Arbeit wird ein numerisches Beispiel besprochen und der Verfasser erörtert dabei eine Methode zur Schrankenverbesserung. Er liefert zu diesem Verfahren einen Konvergenzbeweis, dessen Grundgedanke darauf basiert, dass für die Summe der Fehlerquadrate monotones Abnehmen bewiesen wird. *P. Funk* (Wien).

Karle, J., and Hauptman, H. A note on the solution of the structure-factor equations. Acta Cryst. 4, 188–189 (1951).

Some general elimination theory is sketched to provide a theoretically possible but admittedly impractical solution for the system of algebraic equations to which the equations in question reduce under a change of variables. In the one-dimensional special case given as an illustration a simplifying departure from the outlined procedure is made. *R. Church* (Monterey, Calif.).

Nikolaeva, M. V. On approximate evaluation of oscillating integrals. Trudy Mat. Inst. Steklov. 28, 26–32 (1949). (Russian)

The problem is to calculate $\Phi(\alpha)=\int_a^b f(x)\varphi(\alpha, x)dx$ where α is large and $\varphi(\alpha, x)$ is $\sin \alpha x$, $\cos \alpha x$ or $e^{(\gamma+i\alpha x)}$. If $a=0$, $b=1$ and $f(x)$ is a polynomial of degree n, $\Phi(\alpha)=\sum_{k=0}^n A_k f(k/n)$ where $A_k=\sum_{j=0}^n c_j N_j$ (the constants c_j are given for $n=0, 1, 2, 3$) and $N_j=\int_0^1 x^j\varphi(\alpha, x)dx$. Dividing the interval from a to b into m equal sub-intervals and applying a suitable modification of the above to each yields approximate formulas for $\Phi(\alpha)$ which are given for $n=0, 1, 2$. They correspond to the usual rectangular, trapezoidal, and Simpson quadrature formulas. It is recommended that m be chosen as an even number which would give $\int_a^b f(x)dx$ to the required accuracy by using the corresponding quadrature formula. A limit to the error is to be inferred by also approximating $\Phi(\alpha)$ using $\frac{1}{2}m$. The method is illustrated with $f(x)=e^x$; $a=0$, $b=2$; $n=2$; $m=2, 1$; $\varphi(\alpha, x)=\sin \alpha x, \cos \alpha x$; $\alpha=2, 10, 50, 100$. *R. Church* (Monterey, Calif.).

Gombás, P., and Gáspár, R. Solution of the Thomas-Fermi-Dirac equation. Nature 168, 122 (1951).

The Thomas-Fermi-Dirac equation

$$\psi''=x[(\psi/x)^{\frac{1}{2}}+\beta]^3 \tag{1}$$

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was solved by Umeda [J. Fac. Sci. Hokkaido Imp. Univ. Ser. II. 3, 171–244 (1942); 3, 245 (1949)] for constants β corresponding to the atomic numbers of all neutral atoms and for the boundary conditions

$$\psi(0)=1,\quad \psi(x_0)=0,\quad \psi'(x_0)=0,\quad x_0=\text{border of atom}, \tag{2}$$

resulting from a minimisation of the energy of a single electron. On the other hand, adopting the minimisation of the total energy [Jensen, Z. Physik 93, 232–235 (1935)] the authors reach the boundary conditions

$$\psi(0)=1,\quad \psi(x_0)=\beta^2x_0/16,\quad x_0\psi'(x_0)-\psi(x_0)=0. \tag{3}$$

They report on their computations of the solution $\psi(x)$ of (1) and (3) based on Umeda's ms. tabulations of the solution $\psi_u(x)$ of (1) and (2). Writing $\psi(x)=\psi_u(x)+k\epsilon(x)$, they find a first order approximation to the equation for $\epsilon(x)$ in the form

$$\epsilon''=1.5[(\psi_u/x)^{\frac{1}{2}}+\beta(x/\psi_u)^{\frac{1}{2}}]^2$$

which they solve as an initial value problem with $\epsilon(0)=0$, $\epsilon'(0)=1$, and finally determine k and x_0 from (3). The calculations were carried out for values of β corresponding to neon, argon, krypton, xenon and radon. *H. O. Hartley* (London).

Gardiner, J. G. Integration of the Cowling stellar model. Monthly Not. Roy. Astr. Soc. **111**, 94–101 (1951).

The problem in question requires the numerical solution of a system of ordinary differential equations. The equations themselves present no particular difficulty for numerical integration but the complexity lies in the boundary conditions, which impose certain restrictions at the center of the star, at the outside spherical surface, and also on the internal spherical surface, initially of unknown radius, separating the central convective core and the outer region in radiative equilibrium. The system to be solved is different in the two regions but the solutions and their first derivatives are to be continuous across the separating boundary. The author carries out the calculations and compares results with various earlier solutions *W. E. Milne.*

*****Lanczos, C.** **An iteration method for the solution of the eigenvalue problem of linear differential and integral operators.** Proceedings of the Symposium on Spectral Theory and Differential Problems, pp. 301–316. Oklahoma Agricultural and Mechanical College, Stillwater, Okla., 1951. $3.00.

The author proposes to solve the Fredholm equation (*) $(I-\lambda A)y=b$, for a symmetric operator A, by orthogonalising the sequence $A^n b$ $(n=0, 1, \cdots)$. The orthogonalised sequence b_k is determined by $b_0=b$, $b_{k+1}=Ab_k-\alpha_k b_k-\beta_k b_{k-1}$, where $\alpha_k=\rho_k/\omega_k$, $\beta_k=\omega_k/\omega_{k-1}$, $\rho_k=(Ab_k, b_k)$, $\omega_k=(b_k, b_k)$. The b_k form a coordinate system in which the matrix of A has zero elements except on the main diagonal and the two adjacent diagonals, and a solution of (*) can be written down explicitly in terms of two sets of polynomials satisfying the recurrence relations $p_{k+1}(x)=(x-\alpha_k)p_k(x)-\beta_k p_{k-1}(x)$. Approximate eigenvalues and eigenvectors can be found by using a finite section of the matrix of A, provided the initial vector b is suitably chosen. The method can be extended to eigenvalue problems for differential operators; the initial vector b and the iterates $A^n b$ (up to the order of the approximation) must then be chosen to satisfy the boundary conditions. *G. E. H. Reuter* (Manchester).

Ackerman, Sumner. Precise solutions of linear simultaneous equations using a low cost analog. Rev. Sci. Instruments **22**, 746–748 (1951).

An electrical device for producing the sum of the squares of the errors in a system of linear equations is described. The author advocates the use of this device to obtain an approximate inverse matrix and refining the latter using desk calculators by an unpublished method of O. L. Bowie. *F. J. Murray* (New York, N. Y.).

Bückner, Hans. Ein neuer Typ einer Integrieranlage zur Behandlung von Differentialgleichungen. Arch. Math. **2** (1949–1950), 424–433 (1951).

This paper describes an integrator based on a gear shift type variable speed drive. The gear ratios .1, .3 and .9 can be combined by clutches, reversing gears, and differentials to yield the ratios, at steps of .1, between −1.3 and 1.3. Two figure accuracy can be obtained by combining two such arrangements provided one is driven at one tenth the rate of the other. The clutches are electrically controlled by a punched tape device. The integration is to be used in a differential analyzer in which connections are to be made by a synchro motor system. *F. J. Murray.*

✓*****Synthesis of Electronic Computing and Control Circuits.** By the Staff of the Computation Laboratory. Harvard University Press, Cambridge, Mass., 1951. viii+278 pp. $8.00.

The computing and control circuits discussed in this book are relevant to digital rather than continuous-variable computation. Such circuits, as is appropriate to the on-off switching employed in devices of this type, are functions of several variables in which both the dependent variable and the independent variable are two-valued. The authors set themselves the problem of rationally analyzing and synthesizing vacuum tube switching circuits which have previously been treated in cut-and-try fashion. Vacuum tube operators are introduced for triodes, pentodes, double-triodes, cathode followers, etc., and by means of a normal form for combinations of this sort it is shown how quite different configurations of vacuum tube switches are functionally equivalent. In the next chapter a systematic listing is given for all the $2^{2^3}=256$ switching functions of three variables, first reduced to 22 by reasons of symmetry. With each there is further listed a definite vacuum tube combination generating that switching function; this realization is indeed believed to be the one which is most economical in terms of the number of control grids involved. In an appendix of 48 pages a corresponding treatment is given for the $2^{2^4}=65536$ switching functions of four variables. Other chapters in the book discuss minimizing charts, triggers, rings, "time-variables", rectifiers, coding systems, adders, and multipliers. The character of the subject naturally leads to evident similarities with Boolean algebra. Applications to the practical problems of designing switching systems for digital computers are never far from the scene; the mathematics is never complicated, but the results are nevertheless likely to have very considerable value in simplifying and minimizing the design of intricate switching combinations.

H. Wallman (Gothenburg).

Price, Robert. An FM-AM multiplier of high accuracy and wide range. Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge, Mass., Tech. Rep. No. **213**, i+15 pp. (1951).

Pleskot, Václav. A contribution to anamorphosis. Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodověd. **1949**, no. 4, 17 pp. (1949). (Czech. French summary)

According to the theorem of T.-H. Gronwall [J. Math. Pures Appl. (6) **8**, 59–102 (1912)] there exists a one-to-one correspondence between the families of projectively related Massau determinants equivalent to $F(x, y, z)=0$ and integrals of a certain system of two second-order partial differential equations. After a modified proof of this the author establishes the further special result: any two representations of $F(x, y, z)=0$ as an alignment diagram of genus one (one curved scale and two straight) are projectively related. The end of the introductory paragraph of Gronwall's paper indicates that this was to have been shown in a later paper. *R. Church* (Monterey, Calif.).

Jecklin, H., und Zimmermann, H. Weitere Ergänzungen zur *F*-Methode der Reserveberechnung. Mitt. Verein. Schweiz. Versich.-Math. **51**, 137–163 (1951).

The authors elaborate on their method for computing the premium reserves by hyperbolic interpolation [same Mitt. **50**, 179–196 (1950); **51**, 37–52 (1951); these Rev. **12**, 447,

863] and propose to divide the interval of interpolation into several segments. A different approximating hyperbola is then determined for each segment. This modification is of some practical importance since it permits one to apply the approximation also to reserves with high terminal age.
E. Lukacs (Washington, D. C.).

Zimmermann, Hans. Nomogramme zur "F-Methode." Mitt. Verein. Schweiz. Versich.-Math. **51**, 164–170 (1951).

The author constructs nomograms for the determination of the premium reserve by means of the F-method [see preceding review and the references given there].
E. Lukacs (Washington, D. C.).

ASTRONOMY

Ghosh, N. L. Equilibrium of rotating fluids under the quadratic law of stratifications and the existence of equatorial accl. Proc. Nat. Inst. Sci. India **17**, 391–401 (1951).

The author attempts to explain the observed equatorial acceleration in some celestial bodies by means of very artificial models. The law of density stratification in a rotating fluid mass is assumed to have the form

$$\rho=\rho_0(1-\lambda r^2/a^2-\mu z^2/c^2)$$

or even $\rho=\rho_0(1-\mu z^2/c^2)$, where a and c are semiaxes of the boundary spheroid and λ, μ two parameters. The numerous other investigations of these phenomena seem to have escaped the author's attention
W. S. Jardetzky.

Mal, Babboo, and Behari, Ram. Some formulae of spherical astronomy obtained by tensor method. Proc. Nat. Inst. Sci. India **17**, 271–273 (1951).

The method of tensors was employed by H. and B. S. Jeffreys [Methods of Mathematical Physics, 2d ed., Cambridge, 1950; these Rev. 12, 12] to obtain the effects of geocentric parallax on the right ascension and declination of the moon or a planet. In this paper the method is used to obtain expressions for the changes in longitude, latitude, right ascension and declination of a star due to (1) annual parallax, and (2) aberration.
D. Brouwer.

Kulaschko, B. Zur Verbesserung der Hypothesen für die Dreiecksflächen bei der Parabel. Astr. Nachr. **279**, 213–216 (1951).

Positions at three dates, t, t', t'', in a parabolic orbit are considered. Given the radii vectors r, r'' and the chord k connecting the first and third positions, the author gives formulae that connect the true anomalies v, v'' with these known data. The true anomaly v' for the middle date is computed from v, v'' and the time intervals. All quantities necessary for a direct computation of the ratio of the triangles, $n/n''=r''\sin(v''-v')/r\sin(v'-v)$, are then available.
D. Brouwer (New Haven, Conn.).

Goldsbrough, G. R. The stability of Saturn's rings. Philos. Trans. Roy. Soc. London. Ser. A. **244**, 1–17 (1951).

J. C. Maxwell's memoir on Saturn's rings, published in 1857 [Scientific papers, vol. 1, Cambridge, 1890, pp. 288–376] deals extensively with the motion of a single ring of particles subject to the gravitational attraction by the primary and of each other. In the present article the problem of two rings, treated only incompletely by Maxwell, is examined afresh. The number of particles in each ring is assumed to be the same, but the author remarks that the analysis could be extended to apply to unequal numbers of particles in the two rings.

The discussion proceeds in two steps. It is first shown that if the particles are sufficiently small in mass compared with that of Saturn and sufficiently numerous, there exists a system of steady motions in which the particles are equally spaced on their respective rings and the rings rotate about the primary with appropriate velocities. Small displacements are then given to the particles and it is deduced that the motion can be expressed in terms of series of periodic functions. Hence, in the sense of small displacements the motion is stable. It is further remarked that the problem could be extended to a number of rings and that it would be safe to conclude that, just as for two rings, the motion could be shown to be stable.
D. Brouwer.

Fesenkov, V. G. A criterion of tidal stability and its application in cosmogony. Akad. Nauk SSSR. Astr. Žurnal **28**, 492–517 (1951). (Russian)

The author derives criteria for stability of a cluster (neglecting collisions) and of a fluid mass revolving at some distance about a central mass. These are applied to provide an explanation of certain cosmological problems such as the accretion of mass, the observed average distance of stars in the sun's neighborhood, and the structure of the solar system. In place of Bode's Law the author proposes a formula involving the planets' masses.
R. G. Langebartel.

Kushwaha, R. S. Stability of two stellar models with variable Γ. Proc. Nat. Inst. Sci. India **17**, 323–329 (1951).

The author investigates the stability of the fundamental mode under a small radial adiabatic deformation for the Roche model and the homogeneous model with a central point mass. The law of variation of Γ, the effective ratio of specific heats, is assumed to be $\Gamma=\Gamma_0(1-Ar_0^3/R^3)$ where Γ_0 is the value of Γ at the center, A is a constant, r_0 the distance from the center, and R the star's radius. It is found that in both these models instability sets in when $\bar{\Gamma}$, the average value of Γ, has the value of about 4/3. From a comparison of his results on these models the author concludes that abrupt changes in Γ makes the star more unstable.
R. G. Langebartel (Urbana, Ill.).

***Lindblad, Bertil. An approach to the dynamics of stellar systems.** Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950, pp. 387–402. University of California Press, Berkeley and Los Angeles, 1951. $11.00.

This paper is essentially a restatement of the results contained in an earlier paper by the author [Stockholms Observatoriums Annaler 16, 1–34 (1950); these Rev. 12, 754].
S. Chandrasekhar (Williams Bay, Wis.).

von Weizsäcker, C. F. The evolution of galaxies and stars. Astrophys. J. **114**, 165–186 (1951).

In this paper the author presents a systematic account of his ideas on the evolution of galaxies and stars. The novelty of the approach consists in the crucial roles which are assigned to turbulence, rotation and gravitational instability. As an introduction to the theory, the Kolmogoroff spectrum of turbulence is derived in the form due to the author [Z.

Physik 124, 614–627 (1948); these Rev. 11, 63], namely, that the average relative velocity v of two points separated by a distance l varies as $l^{\frac{1}{3}}$. On the assumption that a compressible fluid will "disintegrate" into a hierarchy of clouds when the relative velocities become comparable to the velocity of sound, the author generalizes Kolmogoroff's law to $v \propto l^{\frac{1}{3}+k}$ where k is a number depending on the ratio of the root mean square velocity of turbulence to the velocity of sound. In cosmogony, the author's basic idea is that an infinite homogeneous turbulent medium will first break up into gravitationally stable turbulent clouds by the action of gravitational instability; then each cloud will become a flattened rotating disk and later dissolve into a uniformly rotating central part, the remaining part returning to cosmic space [cf. Z. Naturforschung 3a, 524–539 (1948); these Rev. 10, 488].

The galaxies are supposed to have been formed as a result of a "competition" between expansion ($v \propto l$) and turbulence ($v \propto l^{\frac{1}{3}}$). And considering the evolution of a galaxy, the author attributes the spiral structure to the distortion of turbulent clouds by non-uniform rotation; and the formation of "bars" is accounted for in terms of its lower gravitational potential relative to a disk.

On the question of the origin of stars, the author's views are: "Stars must be formed rotating because of the turbulence of the original clouds. They lose their rotation probably by a combined magnetic-hydrodynamic process. Both formation and loss of rotation provide gaseous disks in which planets and double stars can be formed". Various other problems such as the "rejuvenation" of stars by accretion and the structure of giant stars are also considered.

S. Chandrasekhar (Williams Bay, Wis.).

Lyttkens, Ejnar. Problems of dark nebulae, treated by the method of moments. Nova Acta Soc. Sci. Upsaliensis (4) 15, no. 2, 89 pp. (1951).

The problem considered in this paper can be formulated as follows: In a "clear" region of the Milky Way the number of stars $A(m)$ with a given apparent magnitude m per unit magnitude interval and per unit solid angle is related to a "density function" $n(z)$ and a "luminosity function" $\Phi(m)$ by the equation of stellar statistics,

$$(1)\qquad A(m)=\int_{-\infty}^{+\infty} n(z)\Phi(m-z)dz.$$

This equation is generally used to determine $n(z)$ on the assumption that $\Phi(m)$ is known. Now if there is an absorbing cloud in the line of sight, then the corresponding frequency function $A_0(m)$ in that direction will be given by

$$(2)\qquad A_0(m)=\int_{-\infty}^{+\infty} n(z)\Phi(m-z-a(z))dz,$$

where $a(z)$ is the absorption in magnitudes produced by the cloud at the "distance" z. The problem in the case of equation (2) is to determine the function $a(z)$ on the assumption that $n(z)$ and $\Phi(m)$ are both known. Several methods based on a numerical inversion of the integral equation (2) have been described in the literature. The method proposed in this paper is to relate $a(z)$ to the moments of $n(z)$ and $\Phi(m)$ by considering

$$(3)\qquad S_j=\int_{-\infty}^{+\infty} m^j\{A(m-\epsilon)-A_0(m)\}dm,$$

where $\epsilon=a(\infty)$. With $A(m)$ and $A_0(m)$ given by equations (1) and (2), S_j can be reduced to the form

$$(4)\qquad S_j=\sum_{k=1}^{j}\binom{j}{k}\nu_{j-k}\int_{-\infty}^{+\infty}\{[z+\epsilon]^k-[z+a(z)]^k\}n(z)dz,$$

where ν_j denotes the jth moment of $\Phi(m)$ and $\binom{j}{k}$ denotes the binomial coefficient. Various special cases of formula (4) are considered and methods are outlined for deriving information concerning $a(z)$ from observations.

S. Chandrasekhar (Williams Bay, Wis.).

Armellini, Giuseppe. L'espansione dell'universo nella meccanica classica. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 8, 15–20 (1950).

The motion according to classical mechanics of a mass of meteorites under Newtonian attraction is discussed on the assumptions (i) that the mass remains spherical and homogeneous, (ii) that any two meteorites of the mass separate with a velocity proportional to their distance apart. The mass density is found to satisfy the equation $(d\rho/dt)^2=\rho^{5/3}(C+4\pi f\rho^{1/3})$, where C is an arbitrary constant and f is the universal gravitation constant. The result is applied to the actual universe, for which C is taken to be 3.37×10^{-13} in c.g.s. units, and it is found that the expansion of the universe would have begun about 1.6×10^9 years ago, which the author points out to be in disagreement with geological researches.

A. J. McConnell (Dublin).

RELATIVITY

***Lorentz, H. A., Einstein, A., Minkowski, H., and Weyl, H. The principle of relativity.** A collection of original memoirs on the special and general theory of relativity. With notes by A. Sommerfeld. Translated by W. Perrett and G. B. Jeffery. Dover Publications, Inc., New York, N. Y., undated. viii+216 pp. Paperbound $1.50; clothbound $3.50.

This is a photographic reprint of a collection of eleven papers by the authors, originally published by Methuen [London, 1923] and translated from Das Relativitätsprinzip [4th ed., Teubner, Leipzig, 1922].

√***Levi-Civita, Tullio. Le problème des n corps en relativité générale.** Mémor. Sci. Math., no. 116. Gauthier-Villars, Paris, 1950. 111 pp. 800 francs.

This excellent monograph on the n-body problem in the general theory of relativity was prepared about ten years ago, but its appearance now is none the less timely for those who have worried themselves with one or another aspect of the problem. Its major achievements are two: a derivation of the equations of motion of n point masses, free from the subtle errors besetting most of the standard treatments; and a careful discussion of the possible contributions, in the Einsteinian approximation, of the finite size and internal constitution of the bodies involved. While the treatment of this latter topic does not profess to the attainment of complete rigor, it is shown on the basis of plausible hypotheses that the effects of the finite extension of the bodies may be absorbed into the Newtonian terms by a first-order redefinition of their masses.

A short historical preface is followed by a lucid although concise chapter on the fundamental concepts and equations of the general relativity theory of gravitation. Chapter II examines the structure of the differential equations of the

gravitational and motional fields for a system of incoherent masses, and shows that they can in fact be written in normal form, to which the Cauchy-Kowalewski existence theorem applies. This is followed by a careful development of the standard successive approximation method, in steps of relative order $(v/c)^2$, to the stage required to bring out the Einsteinian effects in the motion. After a hurried side-glance at the interior problem, Levi-Civita marches on to the nub of the matter in Chapter V, "Reduction to Ordinary Differential Equations". Here the author avoids the conceptual and computational pitfalls into which De Sitter, Chazy and others—among them, upon occasion, Levi-Civita himself—have plunged. Here too is a reasoned justification for Marcel Brillouin's "Principle of Effacement", whereby the contribution of a body to the gravitational field may be ignored in computing its own equations of motion. The monograph ends with a detailed analysis of the 2-body problem (perihelion advance and motion of the center of mass), into which is incorporated a specific account of the absorption into the Newtonian mass of contributions to the field due to the finite extension of the gravitating bodies. *H. P. Robertson.*

√ ***Gödel, Kurt.** **Rotating universes in general relativity theory.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 1, pp. 175–181. Amer. Math. Soc., Providence, R. I., 1952.

This is a report, mainly without proofs, of the results of investigations on rotating universes. The general-relativity field-equations are taken with the cosmological constant λ, and the energy tensor as $T_{ij}=\rho v_i v_j$. The local angular velocity relative to the "compass of inertia" is taken as represented by the 4-vector

$$\omega^i=\tfrac{1}{6}\epsilon^{iklm}v_k(\partial_m v_l-\partial_l v_m)/\sqrt{(-g)}.$$

The author remarks that any space-time with some $\rho>0$ and time-like v_i defined in it, which satisfies the field-equations and permits of no extension free from singularities, and for which ω^i is everywhere continuous and non-zero, represents a rotating universe, but he confines himself to solutions satisfying three further postulates: (i) the solution is to be spatially homogeneous; (ii) space is to be finite (i.e., the topological space whose points are the world-lines of matter is compact); (iii) ρ is not constant. He then states or obtains numerous results of which the following are typical: In rotating universes the 3-spaces of constant density ρ cannot be orthogonal to the world-lines of matter, from which it follows that for sufficiently great distances there must be more galaxies in one half of the sky than in the other half. The topological connectivity of space must be that of a spherical or elliptic 3-space. A necessary and sufficient condition for a spatially homogeneous universe to rotate is that the local simultaneity of observers moving with the matter be not integrable (i.e. do not define simultaneity in the large). For any value of λ (including zero) there exist ∞^8 rotating solutions satisfying the postulates. *H. S. Ruse* (Leeds).

McVittie, G. C. **Two-colour indices and general relativity.** Monthly Not. Roy. Astr. Soc. 110, 590–594 (1950).

A theoretical study of the energy distribution in the spectra of distant nebulae is made, using the cosmological models of general relativity. It is shown that reasonable agreement can be obtained with the measurements of "two-colour indices" by Stebbins and Whitford [Astrophys. J. 108, 413–428 (1948)]. The author points out that more measurements of two-colour indices are required before definite conclusions can be drawn. *A. E. Schild.*

Lampariello, G. **Intorno alle idee generali della fisica Einsteiniana.** Matematiche, Catania 5, 15–31 (1950); 6, 3–41 (1951).

A set of three expository lectures given in May 1950 at the University of Catania.

Majumdar, N. G. **On thermodynamics of matter in a static field.** Bull. Calcutta Math. Soc. 43, 51–55 (1951).

Using the equation of heat conduction, an alternative derivation is given of Tolman's condition for the thermodynamic equilibrium of a spherically symmetric fluid in a static gravitational field. In this derivation no use is made of the explicit form of Einstein's gravitational field equation. *A. E. Schild* (Pittsburgh, Pa.).

Dirac, P. A. M. **Is there an Æther?** Nature 168, 906–907 (1951).

Becquerel, Jean. **Sur la structure de l'espace-temps et la notion physique du temps dans un champ de gravitation statique.** C. R. Acad. Sci. Paris 233, 590–593 (1951).

The author refutes an objection (by private letter) against an earlier note [same C. R. 232, 1617–1619 (1951); these Rev. 13, 168]. In this earlier note the author used monochromatic wave trains to measure time. He now extends his results to the case of static gravitational fields. *A. E. Schild* (Pittsburgh, Pa.).

Gerretsen, J. C. H. **Les fondements géométriques de la relativité restreinte.** Simon Stevin 28, 98–125 (1951).

Exposé didactique d'une axiomatique de la relativité restreinte, à l'exception de l'électrodynamique. Les cinq premiers axiomes ne sont autres que ceux des espaces ponctuels euclidiens et leur présentation est fortement inspirée de celle donnée par le rapporteur dans un ouvrage élémentaire [Éléments de calcul tensoriel, Colin, Paris, 1950; ces Rev. 11, 542]. Le sixième exprime d'une manière précise le caractère indéfini de la métrique de Minkowski sous la forme suivante: il existe un vecteur e à norme positive, tandis que dans les vecteurs orthogonaux à e et non nuls ont une norme négative. Optique géométrique et dynamique du point sont développées parallèlement, sous forme purement vectorielle (dans l'espace vectoriel associé), de façon à pouvoir suggérer la liaison avec la mécanique broglienne. La partie la plus originale de ce travail consiste dans deux démonstrations vectorielles élégantes de la formule de Compton et de la formule de l'effet Doppler quantique, due à Schrödinger. *A. Lichnerowicz* (Paris).

Leaf, Boris. **The continuum in special relativity.** Physical Rev. (2) 84, 345–350 (1951).

The object of this paper is to explore certain dynamical and thermodynamical consequences of the well-known formulation of the equations of motion of a continuous medium as the vanishing of the divergence of the energy-momentum tensor of special relativity. The author develops a new method, in terms of (i) a unit time vector $\hat{u}_\sigma=(1/c)(dx_\sigma/d\tau)$, τ proper time and x_σ coordinates in Minkowski space-time, (ii) a unit space tensor $\tilde{\delta}_{\sigma\nu}=\delta_{\sigma\nu}+\hat{u}_\sigma\hat{u}_\nu$, where $\delta_{\sigma\nu}$ is the Kronecker delta, for producing new covariant quantities by resolving known covariant tensors into space and time components. In particular, he discusses a

symmetric second order tensor $\psi_{\sigma\tau}=\phi_{\sigma\tau}+\chi_{\sigma\tau}$, where

$$\phi_{\sigma\tau}=\bar{\bar{\delta}}_{\sigma\mu}\bar{\bar{\delta}}_{\tau\nu}\psi_{\mu\nu},\quad \chi_{\sigma\tau}=-(\hat{u}_\sigma\hat{u}_\mu\psi_{\mu\tau}+\hat{u}_\tau\hat{u}_\nu\psi_{\sigma\nu}+\hat{u}_\sigma\hat{u}_\tau\hat{u}_\mu\hat{u}_\nu\psi_{\mu\nu}).$$

Identifying $\psi_{\sigma\tau}$ with the energy-momentum tensor, the equation of motion $\partial_\sigma\psi_{\sigma\tau}=0$, where ∂_σ indicates partial differentiation with respect to x_σ, gives a time component which can be identified as the first law of thermodynamics. The equation obtained on the space components is identified as Newton's second law of dynamics for an element of the continuum, thermal effects included. The second law of thermodynamics is formulated by postulating a stress system $\phi'_{\sigma\tau}$ independent of surroundings, whereas $\phi_{\sigma\tau}$ represents the stress exerted on an element by its surroundings. All the usual thermodynamic variables, except volume, are found to be Lorentz invariant, whereas M. Planck [Ann. Physik (4) **26**(331), 1–34 (1908)] obtained scalar invariant pressure and entropy but not temperature. *G. J. Whitrow.*

Fantappiè, Luigi. Costruzione effettiva di prodotti funzionali relativisticamente invarianti. Ann. Mat. Pura Appl. (4) **29**, 43–69 (1949).

The object of this paper is to construct functional scalar products of two functions (such as are used in quantum mechanics), which are relativistically invariant. This is done by first replacing the ordinary space-time variables in the two functions by spinor variables in the usual manner, and then taking the "functional projective trace" of the product of the two functions thus formed, as already defined by the author [Ann. Mat. Pura Appl. (4) **22**, 181–289 (1943); these Rev. **8**, 589]. The author then proceeds to verify that the functional products thus defined are Lorentz invariant, and also to show how such products can be made hermitean symmetric. *A. J. McConnell* (Dublin).

Drăganu, Mircea. Some observations on general Lorentz transformations. Acad. Repub. Pop. Române. Bul. Şti. Ser. Mat. Fiz. Chim. **2**, 561–565 (1950). (Romanian. Russian and French summaries)

The author writes the equations of the Lorentz transformation in vectorial form, gives the transformation formulas for velocities and for the electro-magnetic field, and derives the formula for the resultant of two velocities.

A. Erdélyi (Pasadena, Calif.).

Mariani, Jean. Électromagnétisme et relativité. Le magnétisme terrestre comme conséquence de la relativité génerale. II. Cahiers de Physique no. **33**, 31–62 (1948).

The author continues his unified theory of gravitation and electricity [Cahiers de Physique, no. **28**, 23–54 (1945); these Rev. **9**, 387] by considering the trajectories of particles in an electromagnetic and gravitational field as the autoparallel curves in a Riemann space with torsion (in Cartan's sense). The equations of these curves are $u_{i,j}u^j=\omega_{ij}u^j$, where u^i is the unit tangent vector, $\omega_{ij}=2\Lambda^k{}_{ij}u_k$ ($\Lambda^k{}_{ij}$ being the torsion tensor), and the comma indicates covariant differentiation with respect to the metric of the Riemann space. The quantities $\Lambda^k{}_{ij}$ are not given uniquely when the electromagnetic field is known but only in the combination $\Lambda^k{}_{ij}u^j=\frac{1}{2}\omega_i{}^k$. The field equations are obtained by equating to zero that part of the contracted Christoffel tensor of the affine connection which is uniquely determined by the electromagnetic field. This, together with considerations of symmetry, leads to the field equations $R_{ij}-\frac{1}{2}Rg_{ij}=-Ru_iu_j+2\omega_i{}^k\omega_{kj}-\frac{1}{2}g_{ij}\omega_{mn}\omega^{mn}$, with $\omega^k{}_{i,k}=\frac{1}{2}Ru_i$, $\omega_{ij,k}+\omega_{jk,i}+\omega_{ki,j}=0$, where R_{ij} is the Ricci tensor of the Riemann metric. The first set of equations are the known Einstein equations for an electrified material fluid in a gravitational and electromagnetic field and the second two sets are Maxwell's equations, provided that we put $R=8\pi\gamma\rho/c^2$, $\omega_{ij}=\pm(\gamma^{\frac{1}{2}}/c)F_{ij}$, $\sigma=\pm\gamma^{\frac{1}{2}}\rho$, where F_{ij} is the electromagnetic skew-symmetric tensor, ρ is the mass density, σ the electric charge density, and γ is the universal gravitation constant. Consequently, these field equations are applicable only to a fluid for which the ratio σ/ρ is a universal constant $\pm\gamma^{\frac{1}{2}}$. *A. J. McConnell.*

Mariani, Jean. Électromagnétisme et relativité. Le magnétisme terrestre comme conséquence de la relativité générale. III. Cahiers de Physique no. **34**, 1–28 (1950).

The author shows that the field equations of his unified field theory [see preceding review] can be deduced from the variational principle $\delta\int L\sqrt{-g}d\tau=0$, where

$$L=\tfrac{1}{2}Rg^{ij}u_iu_j-\tfrac{1}{2}g^{im}g^{jn}(u_{m,n}-u_{n,m})(u_{i,j}-u_{j,i})$$

and we take variations both of the g_{ij} and of the u's. This theory applies only to an electrified fluid whose electric charge density σ is connected with the mass density ρ by the relation $\sigma=\pm\gamma^{\frac{1}{2}}\rho$, where γ is the universal gravitation constant. The author proceeds to apply the theory to a rotating material sphere and concludes that it implies the existence of a magnetic moment equal to $-(\gamma^{\frac{1}{2}}/6c)M$, where M is the angular momentum of the sphere. Applying this result to the earth and taking the angular momentum of the earth to be 5.917×10^{40} c.g.s. units, we find the theoretical value of the earth's magnetic moment to be -8.5×10^{25} e.m.u., whereas the observed value is -8.5×10^{25} e.m.u. A similar calculation is carried out for the sun, but the agreement with observation is not so satifactory. The paper concludes with some remarks on the bearing of the theory on the electric field of the earth. *A. J. McConnell.*

Gião, Antonio. On the origin of positive and negative electricity. Portugaliae Math. **8**, 143–153 (1949).

The difference in the behavior of mass and electric charges is discussed in the light of author's unified field theory [see Physical Rev. (2) **76**, 764–768 (1949); these Rev. **11**, 547].

C. Kikuchi (Upton, N. Y.).

★Synge, John L. The relativity theory of A. N. Whitehead. The Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, College Park, Md., 1951. ii+49 pp. $1.40.

These are notes of three lectures on the Whitehead theory of gravitation, published in 1922 in Whitehead's book, The Principle of Relativity [Cambridge, 1922]. In these lectures, the author is concerned (as he clearly states) not with the philosophical implications but rather with the mathematical structure and the physical meaning of Whitehead's theory. All of the three consequences of Einstein's theory that can be treated by experiment also agree (according to Synge) with Whitehead's theory. As far as perihelion motion and the deflection of light are concerned, the argument is convincing. It is less convincing for the red-shift since Synge's interpretation differs here from Whitehead's original interpretation. The equations for the two-body problem in Whitehead's theory are also formulated but not compared with those in Einstein's theory. *L. Infeld* (Warsaw).

MECHANICS

✶Bilimović, Anton. Racionalna mehanika. I. Mehanika tačke. [Rational mechanics. I. Mechanics of a point]. 2d ed. Naučna Knjiga, Belgrade, 1950. viii+331 pp.

✶Bilimović, Anton. Racionalna mehanika. II. Mehanika sistema. [Rational mechanics. II. Mechanics of a system]. Naučna Knjiga, Belgrade, 1951. vii+405 pp.

✶Fischer, Otto F. Universal Mechanics and Hamilton's Quaternions. Axion Institute, Stockholm, 1951. vi+356 pp. $10.00.

The theory of linear algebras has many applications in mathematical physics. The author attempts to describe these in this book. However, the book is a confusing one for a number of reasons: First, because the author insists in using the word quaternions to describe various different linear algebras. Second, because (as the author states in the preface) "the quaternions are applied in a rather haphazard manner to a series of examples, a wandering together with the reader through many countries". Third, the book is not self-contained. Long stretches of the discussion are applied to theories not well known and not adequately summarized in the book itself.

The first chapter is devoted to a review of Gibb's vector analysis. The second chapter is very extensive and discusses a great variety of fields. Among them are: elasticity, Maxwell electromagnetic theory, special and general relativity and the Dirac electron theory. However, the discussion is not at all clear or complete. For example, the author never discusses the transformation properties of the quantities with which he deals. This leads to mathematical difficulties. One of them is evident on page 67 where it is claimed that the components of a Dirac wave function (a four component simple spinor) are linearly related to the components of two real three-dimensional vectors. In view of the transformation properties of both sets of quantities this equation can only hold in a single frame of reference. In addition it is clear from the theory of spinors that the two three-dimensional vectors are invariantively connected with a symmetric two-index two-component spinor.

Chapter III is entitled "Quadric quaternions and Eddington E numbers". A quadric quaternion is an element of the Clifford algebra of order 16. Clearer presentations of this material may be found in the literature. Chapter IV is entitled "Further developments and ideas". It contains a partial review of H. G. Küssner's [Principia physica, Vandenhoeck and Ruprecht, Göttingen, 1946] attempts to formulate the fundamental physical laws by means of equations involving Stieltjes integrals. The chapter concludes with a dissertation on the importance of quaternions for the understanding and unifying of physical theories.

A. H. Taub (Urbana, Ill.).

✶Maxwell, J. Clerk. Matter and motion. Dover Publications, Inc., New York, N. Y., undated. xiii+163 pp. (1 plate). Paperbound, $1.25; clothbound, $2.50.

Reprinted from an earlier edition [Society for promoting Christian knowledge, London, 1920] with notes and appendices by J. Larmor.

✶Levitskiĭ, N. I. Proektirovanie ploskih mehanizmov s nizšimi parami. [Design of plane mechanisms with lower pairs.] Institut Mašinostroeniya, Akad. Nauk SSSR, Moscow-Leningrad, 1950. 182 pp.

This is a monograph on the method of least squares in approximate linkage design. The method also appears in combination with the method of linear corrections. The author's own results constitute the bulk of the book. The problem is to generate a function or to trace a curve; the mechanisms are the four-hinge linkage and the simple slider-crank linkage. All solutions are followed by numerical illustrations. The presentation is very explicit.

An introductory chapter 1 discusses the method and the mathematical prerequisites. In chapter 2 the four-hinge linkage $ABCD$ is described by means of six parameters (three side ratios, the direction of the stand AD, and the limits of the working-range interval); the relation between the lever angle ψ and the crank angle α is the generated function. Assuming C to be a crosshead C_s and ψ the required function of α, $\Delta_q = BC_s^2 - BC^2$ is the accepted measure of the deviation. It is given the form
$$F(\alpha) - p_0\varphi_0(\alpha) - p_1\varphi_1(\alpha) - p_2\varphi_2(\alpha),$$
and p_0, p_1, p_2 are determined to make $\int \Delta_q^2 d\alpha$ a minimum over the working range. It can also be represented in the form $F(\alpha) - p_0\varphi_0(\alpha) - p_1\varphi_1(\alpha) - p_2\varphi_2(\alpha) - p_3\varphi_3(\alpha) - p_0p_1\varphi_4(\alpha)$, and then p_0, p_1, p_2, p_3 can be determined. The functions F and φ are not uniquely determined, and have been chosen so as to make them dependent on α and the given parameters of the mechanism. Worked-out examples show how linear corrections can refine the approximation still further. The functions generated here are $\psi = \alpha$ and $\psi = \log \alpha$. Also, the "best", i.e. uniform (=Chebyshev: equal extreme deviation) approximation is obtained (as a further refinement) by successive linear corrections. All this work is numerical.

In chapter 3, for curve tracing by means of a connecting-rod point M, C is again replaced by a slider-crank and $\Delta_q = DC_s^2 - l^2$ is accepted as the measure of the deviation when the tentative point M is moved upon the curve to be traced. Δ_q is treated as in the preceding chapter. Numerical examples deal with a ballistic curve and a certain closed curve, both given by means of a finite number of points. The problems of three, four and five parameters are treated, the latter again by applying successive linear corrections.

In chapter 4, the mechanism is a slider-crank linkage ABC and the abscissa X of the crosshead C is a function of the angle α of the crank AB, used to generate the given function. Again if BC is assumed to be variable, $\Delta_q = BC^2 - l^2$ is accepted as a measure of the deviation if X is the given function of α. The same method of least square deviation is used for the determination of two, three, and four parameters. The numerical work deals with the function $X = \tan \alpha$.

Chapter 5 takes up curve tracing by means of a point of the connecting rod of the slider-crank mechanism. Here $\Delta_q = AB^2 - l^2$ (AB − crank) is assumed as the measure of the deviation. The previously considered ballistic curve is used for numerical work.—A bibliography of 100 titles is appended. It contains only seven non-Russian references, of which only one [Svoboda, Computing mechanisms and linkages, McGraw-Hill, New York, 1948; these Rev. 9, 381] is in English.

A. W. Wundheiler (Chicago, Ill.).

Levitskiĭ, N. I. Design of a four-bar linkage with four and five variable parameters, tracing a given trajectory. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 7, no. 27, 5–38 (1949). (Russian)

Levitskiĭ, N. I. Design of a four-bar linkage for a given motion of the follower crank. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 7, no. 28, 26–73 (1949). (Russian)

Levitskiĭ, N. I., and Levitskaya, O. N. Design of slider-crank linkages. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 8, no. 31, 5–46 (1950). (Russian)

These papers have been incorporated in the first author's book reviewed above as pages 121–140, 47–92, and 160–175, respectively. The least-squares method, together with an idea of deviation measurement, are applied, as described in the review of the book. *A. W. Wundheiler.*

Platrier, Charles. Contribution à la théorie du pendule de Foucault. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 37, 762–779 (1951).

Using series expansions derived from the elliptic-integral solution of the spherical pendulum problem, and neglecting certain small terms, the author studies the horizontal projection of the trajectory of a Foucault pendulum for general initial conditions. *P. Franklin* (Cambridge, Mass.).

Breves Filho, J. A. Rigid displacements. Bol. Soc. Mat. São Paulo 3, no. 1-2 (1948), 9–16 (1951). (Portuguese)

Let two given particles of a rigid solid occupy the points O and P in the initial configuration and the points O', P' in the final configuration and let $\Delta P=P'-P$, $\Delta O=O'-O$. For displacements which are not pure translation the author uses cartesian coordinates to establish the existence of an angle of rotation ϵ and a vector of rotation $\mathbf{\Omega}$ such that

$$\Delta P=\Delta O+\tfrac{1}{2}\mathbf{\Omega}\wedge[\mathbf{\Omega}\wedge(P-O)]+\cos\tfrac{1}{2}\epsilon\,\mathbf{\Omega}\wedge(P-O)$$

This fundamental formula is used to establish the properties of the parameters of Euler, and the Cayley-Klein parameters. *L. M. Milne-Thomson* (Greenwich).

Artobolevskiĭ, I. I. A new method for the determination of flywheel masses. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 1, 49–56 (1947). (Russian)

The author first derives the formula

$$\omega_r^2 dI_r/dt=2\sum[mva+I\omega\epsilon]$$

where ω_r is the constant angular speed of the driving member, $I_r\omega_r^2/2$ the kinetic energy of the machine, v and a the speed and tangential acceleration of the mass center of a link, ω and ϵ its angular velocity and acceleration, and m, I its mass and central moment of inertia. Then he suggests a certain correction for the case of variable ω_r, which is based on the extreme values of ω_r and I_r. He argues that his method is superior to the graphical ones of Wittenbauer, Mertsalov, and Radinger. *A. W. Wundheiler* (Chicago, Ill.).

Hill, E. L. Hamilton's principle and the conservation theorems of mathematical physics. Rev. Modern Physics 23, 253–260 (1951).

The object of this paper is to clarify the relationship of the equations of motion and the conservation theorems as they follow from the variational principle. A systematic review of both sets of equations is given and illustrated by the example of classical mechanics and scalar meson field theory. In both cases conservation of momentum energy, angular momentum, and the center of mass conservation theorem are deduced from symmetry transformations. *L. Infeld* (Warsaw).

★**Dugas, René. Genèse, rôle et interprétation des principes variationnels dans les différentes mécaniques.** Congrès International de Philosophie des Sciences, Paris, 1949, vol. III, Philosophie Mathématique, Mécanique, pp. 121–128. Actualités Sci. Ind., no. 1137. Hermann & Cie., Paris, 1951.

Hydrodynamics, Aerodynamics, Acoustics

Truesdell, C. On the equation of the bounding surface. Bull. Tech. Univ. Istanbul 3 (1950), no. 1, 71–78 (1951). (English. Turkish summary)

L'auteur étudie les conditions nécessaires et suffisantes pour que la surface $f(x,t)=0$, soit une surface matérielle ou puisse constituer une frontière, dans un mouvement donné d'un fluide matériel. La théorie des ondes développée par Hugoniot et Hadamard lui permet d'obtenir très simplement les résultats suivants. (1) Si le mouvement est continu, la condition classique, donnée par Lagrange (*) $Df/Dt=0$ (où D/Dt désigne la dérivée par rapport au temps pour une particule matérielle) est nécessaire et suffisante pour que $f=0$ soit une surface matérielle. (2) Quand le mouvement est discontinu sur la surface, la densité ρ devenant nulle ou infinie, les conclusions précédentes doivent être modifiées et l'auteur indique la signification de la condition (*) dans ces deux cas. Les résultats généraux précédents sont illustrés par un exemple de mouvement dans lequel (*) est vérifié par une surface qui n'est ni une surface matérielle ni une surface frontière. *R. Gerber* (Grenoble).

Mayai, Yoshihiro. On the problem of a stationary vortex of a fluid passing the surface of an obtuse angle. J. Jap. Soc. Appl. Mech. 4, 90–95 (1951). (English. Japanese summary)

Two straight walls meet at an obtuse angle ABC. A stream of fluid moves along AB on the outside of the bend. The author solves by conformal tranformation the related problem for the flow of an inviscid incompressible fluid on the assumption that the wall AB is continued to D by a thin flat rigid plate BD, where $BD=l$. He finds that there is a stationary vortex between BD and BC. Making the further assumption that the strength of this vortex is a minimum, numerical values are obtained in terms of l and the streamlines are sketched. Physical reasons are adduced why this solution should approximate to the problem of flow of a real fluid past the bend (with no plate BD). Nevertheless the existence of the plate leads to a pressure difference on its two faces. Some observations are made concerning the value of l. *L. M. Milne-Thomson* (Greenwich).

Müller, W. Längsbewegung eines Rotationskörpers in der Flüssigkeit. Ing.-Arch. 19, 282–295 (1951).

The author considers the flow about bodies of revolution which can be generated by a given source-sink distribution along the axis of symmetry given by $f(x)$, $-1\leqq x\leqq 1$, where $\int_{-1}^{1}f(x)dx=0$. Introducing elliptic coordinates ζ, μ, where $x=\zeta\mu$, $\eta=[(\zeta^2-1)(1-\mu^2)]^{\frac{1}{2}}$ = radial distance from the symmetry axis, he obtains a well known formula [cf. Kaplan,

Tech. Rep. Nat. Adv. Comm. Aeronaut. no. **516** (1935), eq. (15)] for the meridian of the body

$$\frac{A}{\pi v_1}\sum\frac{\alpha_\nu}{\nu(\nu+1)}P_\nu'(\mu)Q_\nu'(\zeta)+1=0,$$

where $\alpha_\nu=\frac{1}{2}(2\nu+1)\int_{-1}^{1}f(t)P_\nu(t)dt$ and P_ν and Q_ν are Legendre functions. The velocity components and pressure coefficient are expressed in terms of $\bar{P}_\nu=(1-\mu^2)P_\nu'$ and $\bar{Q}_\nu=-(\zeta^2-1)Q_\nu'$. A table of $\bar{Q}_\nu(\zeta)$, $\nu=1(1)5$, $\zeta=1(.001)1.050$, 4S is included. Special forms for $f(t)$ are treated in the second half of the paper, including functions

$$f(t)=c_0(t+1)^n+\cdots-c_{n-1}(t+1),\quad c_0=\pm1,$$

and some relevant values of the α_ν are tabulated. There are a number of figures showing the results for special choices. Except in some details the method does not go beyond Kaplan's first method [loc. cit.] and the author does not treat the mathematical problem of finding $f(t)$ when the meridian of the body is given. *J. V. Wehausen.*

Sretenskiĭ, L. N. The oscillation of a fluid in a movable basin. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1951**, 1483–1494 (1951). (Russian)

The author considers a movable rectangular basin subject to a restoring force proportional to the displacement and filled with a liquid to some fixed depth h. First, the motion of the liquid is determined by assuming a slow harmonic motion for the basin and the linearized theory of gravity waves. From this the force exerted by the liquid upon the basin walls for a given frequency may be computed. The equation of motion of the basin, taking account of this force as well as the restoring force, now leads to an equation for the frequency. This rather complicated equation is studied in some detail. A numerical calculation with selected parameters is carried out for the case when the basin is the weight of a pendulum. *J. V. Wehausen* (Providence, R. I.).

Nagamatsu, H. T. Circular cylinder and flat plate airfoil in a flow field with parabolic velocity distribution. J. Math. Physics **30**, 131–139 (1951).

The problem is the two-dimensional incompressible inviscid flow over a body with non-uniform parallel flow at infinity. By assuming small non-uniformity of the flow at infinity and small disturbance due to the presence of the body, the author reduces the problem to that of solving a Laplace equation with specified velocity at the solid boundary derived from the undisturbed flow. The particular cases computed are circular cylinder in "slightly" parabolic flow with or without circulation, and flat plate airfoil in "slightly" parabolic flow or linear shear flow. It seems to the reviewer that the deviations of stream lines from the parallel flow created by the presence of a circular cylinder is certainly not small. Therefore it is unlikely that the author's result for this case can be valid. For the flat plate airfoil, the same reasoning would show that the author's solution is not applicable for points near the leading edge. *H. Tsien.*

Funaioli, Ettore. Sul calcolo di schiere alari di profili sottili di curvatura non trascurabile. Pont. Acad. Sci. Acta **14** (1950), 45–55 (1951). (Italian. Latin summary)

A method of calculation of flows through cascades of thin aerofoils whose center lines approximate to circular arcs is given in principle, without details of computational procedures. *M. J. Lighthill* (Manchester).

Moisil, Gr. C. La méthode des fonctions de variable hypercomplexe dans l'hydrodynamique plane des liquides visqueux incompressibles. Acad. Repub. Pop. Române. Stud. Cerc. Mat. **1**, 9–39 (1950). (Romanian. Russian and French summaries)

It is well known that in the theory of steady, incompressible, irrotational, plane flow of an ideal fluid, analytic functions $u(x, y)-iv(x, y)$ of the complex variable $z=x+iy$, where $i^2+1=0$, play an important rôle, inasmuch as any such function yields a possible flow of this kind, with u and v the velocity components in the x and y directions. L. Sobrero [Mem. Atti Accad. Italia Mem. Cl. Sci. Fis. Mat. Nat. **6**, 1–64 (1934); Theorie der ebenen Elastizität unter Benutzung eines Systems hyperkomplexer Zahlen, Teubner, Leipzig, 1934] has shown a similar connection between the theory of plane elasticity and the theory of analytic functions $a(x, y)+jb(x, y)+j^2c(x, y)+j^3d(x, y)$ of the hypercomplex variable $x+jy$, where $(1+j^2)^2=0$. The author shows a similar connection between analytic functions in the sense of Sobrero and the theory of slow, steady, incompressible, plane flow of a viscous fluid. For example, any analytic function of $x+jy$,

$$F=-\Psi+j(\Phi-4\mu v)+j^2(\Psi+4\mu u)-j^3\Phi,$$

represents such a plane flow, with body forces absent; and

$$\frac{dF}{dz}=\sigma_y+j(\tau-2\mu\omega)-j^2\sigma_x+j^3\tau,$$

where u and v are the velocity components in the x and y directions; σ_x, σ_y, τ are the stress components; μ is the coefficient of viscosity; and $\omega=\partial v/\partial x-\partial u/\partial y$ is the vorticity. Several examples of plane viscous flows are worked out in detail. *J. B. Diaz* (College Park, Md.).

Berker, Ratip. Sur l'impossibilité pour un fluide visqueux homogène ou hétérogène d'un mouvement à la Poinsot. Bull. Tech. Univ. Istanbul **3** (1950), no. 1, 61–66 (1951). (French. Turkish summary)

The author considers a mass of fluid, the particles of which attract one another according to the law of Newton. Extending results of Poincaré and Appell to include the cases of a viscous fluid (1) with constant density, (2) with density a function of pressure only, (3) heterogeneous, he shows that any rigid body motion of the fluid is necessarily a steady rotation about a fixed axis, if in case (3) an exceptional distribution is excluded. The possibility of making this extension to cover viscosity is due to the fact that viscous stresses do not occur in a rigid body motion.

J. L. Synge (Dublin).

Hasimoto, Hidenori. Note on Rayleigh's problem for a bent flat plate. J. Phys. Soc. Japan **6**, 400–401 (1951).

The author considers the same problem as was recently treated by Sowerby [Philos. Mag. (7) **42**, 176–187 (1951); these Rev. **12**, 871], namely, the flow starting from rest of a viscous incompressible fluid bounded by two infinite intersecting planes making an angle α which suddenly start to move with constant velocity W. However, whereas Sowerby considered only $\alpha=\pi/n$, n a positive integer, the author allows $0<\alpha\leqq2\pi$. The local friction coefficient is obtained as a series involving confluent hypergeometric functions and the friction force per unit length is given explicitly for various multiples of $\pi/4$ and $\pi/6$ between $\pi/2$ and 2π. The results agree, as the author points out, with Sowerby's where they overlap. The method is only sketched.

J. V. Wehausen (Providence, R. I.).

Yamada, Hikoji. A method of approximate integration of the laminar boundary layer equation. Rep. Res. Inst. Fluid Eng. Kyushu Univ. 6, no. 2, 87–98 (1950).

It is proposed to improve the von Kármán-Pohlhausen approximation by using polynomials of degrees higher than the fourth for the velocity profile and satisfying integral conditions other than just the momentum-integral equation. The additional equations are obtained by calculating higher moments of the longitudinal momentum component. For the special case of sixth-degree profile, using two moment-equations in addition to the usual momentum (zeroth moment) equation, the author works out the method in detail and provides tables of the required coefficients. The approximation obtained by using a fifth-degree profile and a total of two moment-equations is applied to a number of test cases where accurate solutions are known: Blasius' (flat-plate) case, two of the Falkner-Skan-Hartree cases including incipient separation, linearly-retarded flow, Schubauer's elliptic cylinder, and Hiemenz's circular cylinder. In every case improvement over the Pohlhausen method is noted, and in some cases the agreement with more exact calculation is good. *W. R. Sears* (Ithaca, N. Y.).

Loitsianskii, L. G., and Bolshakov, V. P. On motion of fluid in boundary layer near line of intersection of two planes. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1308**, 27 pp. (1951).

Translated from Trudy Central. Aero-Gidrodinam. Inst. no. **279** (1936).

Brown, W. Byron, and Donoughe, Patrick L. Tables of exact laminar-boundary-layer solutions when the wall is porous and fluid properties are variable. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2479**, 68 pp. (1951).

Fifty-eight new exact solutions of the laminar boundary layer equations for negligible Mach number are computed. In each case the main stream velocity distribution is proportional to x^{Eu}, where x is measured from the start of the boundary layer and the "Euler number" Eu is interpreted as a pressure gradient parameter

$$-\frac{x}{l_\infty u_\infty^2}\frac{dp}{dx},$$

where suffix infinity refers to the main stream. The wall temperature T_w is supposed uniform, but the ratio T_w/T_∞ is allowed large variation, so that compressibility has to be allowed for and the velocity and temperature equations become interconnected. The viscosity, thermal conductivity and specific heat at constant pressure are supposed to vary as the absolute temperature to the powers 0.7, 0.85 and 0.19 respectively. The Prandtl number at the wall is taken to be 0.7.

The ratio T_∞/T_w is given the values 4, 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, and Eu is in each case given a range of values from $+1$ down to the small negative value corresponding to separation. This value appears to vary approximately in proportion to $(T_\infty/T_w)^{\frac{1}{2}}$. In each case the profiles of the stream function and its first three derivates, the temperature and its first two derivatives, and the velocity, are tabulated. Finally there is a grand table of Nusselt number Nu, skin friction coefficient C_f, displacement momentum and convection thicknesses, for the different values of Eu and T_∞/T_w. In every case the spread of values Nu and C_f with Eu is widened by wall heating and narrowed by wall cooling. For accelerated flow with wall heating, the velocity in the boundary layer sometimes exceeds that in the main stream (and the momentum thickness can be negative). The explanation of these facts, namely that (e.g. with wall heating) the pressure gradient acts on fluid of reduced density, is not clearly stated.

Further calculations are made on the assumption that coolant air at temperature T_w is introduced through pores in the wall. No clear statement is made as to the velocity of the introduced coolant for which the calculations have been done, and its mode of variation with x, but this appears to be firstly

$$(\mathrm{Eu}+\tfrac{1}{2})(\nu_w U_\infty/x)^{\frac{1}{2}}$$

and secondly half this value. In each of these cases, T_∞/T_w is given the values 1, 2, 4, and the properties of the boundary layers are tabulated as before. *M. J. Lighthill.*

Okabe, Jun-ichi. The vorticity in the laminar boundary layer of a surface of revolution. Rep. Res. Inst. Fluid. Eng. Kyushu Univ. 6, no. 2, 47–51 (1950).

Several authors have attacked the plane-flow problem of a viscous fluid by writing the velocity as $\mathbf{v}=\mathbf{v}_0+\mathbf{v}_1$, where $\mathbf{v}_0$ denotes the potential flow about the same obstacle. A transformation attributed to Boussinesq then reduces the linearized equation for the vorticity to the form of the heat-conduction equation. This is an attempt to extend the same method to axisymmetric flows. Instead of the coordinates ϕ, ψ (velocity potential and stream function of $\mathbf{v}_0$) used in Boussinesq's transformation, the author employs $\int r_s^2(\phi)d\phi$ and ψ, where r_s is the cylindrical radius of the solid body surface and now ψ is Stokes' stream function for the irrotational flow. For the region quite near the body surface, the heat-conduction equation is again obtained for a quantity related to the vorticity. Solutions similar to those used by Burgers [Müller, Einführung in die Theorie der zähen Flüssigkeiten Akademische Verlagsgesellschaft, Leipzig, 1932, pp. 188–193] and by Preston [Philos. Mag. (7) **31**, 413–424 (1941); these Rev. **3**, 22] are written down. The author argues that the two solutions must be the same, except for higher-order terms. He also states that he has applied this theory to estimate the separation point of flow about a sphere. *W. R. Sears* (Ithaca, N. Y.).

Howarth, L. The boundary layer in three dimensional flow. II. The flow near a stagnation point. Philos. Mag. (7) **42**, 1433–1440 (1951).

The general equations for three-dimensional boundary-layer flow given by the same author [Philos. Mag. (7) **42**, 239–243 (1951); these Rev. **12**, 871] are applied to flow at a stagnation point of the exterior (potential) flow occurring at a regular point of the body surface. The regularity leads to neglect of some curvature terms, so that the equations are the same as if the surface were plane, and ultimately to the choice of Cartesian coordinates x, y, in the surface and ζ normal to it, such that the exterior flow velocity components are ax and by in the x and y directions, respectively.

The boundary-layer velocity components are then

$$u=axf'(z),\quad v=byg'(z),\quad w=-[\nu/a]^{\frac{1}{2}}[af(z)+bg(z)],$$

where $z=[a/\nu]^{\frac{1}{2}}\zeta$, ν is the kinematic viscosity, and (with $c=b/a$)

$$f'^2-ff''-cgf''=1+f'''$$
$$g'^2-gg''-c^{-1}fg''=1+c^{-1}g'''.$$

Now, $c=0$ represents two-dimensional flow, and $c=1$ axisymmetric flow, near stagnation points; the former was studied by several authors, the latter by Homann [Z.

Angew. Math. Mech. 16, 153–164 (1936)]. Here calculations are carried out for $0<c<1$, which includes all interesting cases.

First, formulas for expansion in powers of c are proposed, and for this purpose tabular data are given which yield $f'(z)$ and $g'(z)$ up to the terms in c^2. Direct numerical integration of the differential equations displayed above has also been carried out for $c=0.25$, 0.50, 0.75 and 1.0. Tables of values are given for $f'(z)$, $g'(z)$, $f''(0)$, $g''(0)$, $f(z)$, and $g(z)$, for these values of c. Finally, some features of the layers, such as direction of flow near the surface, are pointed out.

W. R. Sears (Ithaca, N. Y.).

Howarth, L. Note on the boundary layer on a rotating sphere. Philos. Mag. (7) 42, 1308–1315 (1951).

The problem considered is that of the laminar boundary layer on a steadily rotating sphere in a fluid otherwise at rest. The author's general three-dimensional boundary-layer equations [Philos. Mag. (7) 42, 239–243 (1951); these Rev. 12, 871] are written for this case in the customary spherical coordinates r, θ, ϕ. The flow near the pole must be similar to that over a rotating disc, which was studied by von Kármán [Z. Angew. Math. Mech. 1, 233–252 (1921), pp. 244–247] and Cochran [Proc. Cambridge Philos. Soc. 30, 365–375 (1934)]. There is inflow there, while at the equator the layers of the two hemispheres collide and there is outflow. The boundary-layer equations are not capable of describing this equatorial situation. After discussing briefly the possibilities of a series solution, the author proceeds to an approximate solution by the momentum-integral method, following von Kármán (loc. cit.) in the choice of functions for the boundary-layer profiles of the flow components in the θ and ϕ directions. The approximation obtained indicates that the profiles have the same forms everywhere; these profiles are tabulated. Also tabulated are boundary layer thickness as function of θ, and other results.

In an appendix the author considers another, unrelated problem; namely, the flow along a thin hollow semi-infinite cylinder rotating about its axis and placed axisymmetrically in a uniform stream. Here the boundary-layer equations are especially simple and are solved by means of the well-known flat-plate (Blasius) function. The drag of the cylinder is unaffected by its rotation; the frictional moment is calculated.

W. R. Sears (Ithaca, N. Y.).

***Foundations, aerodynamics of high speed. With a bibliography compiled by George F. Carrier.** Dover Publications, Inc., New York, N. Y., 1951. x+286 pp. Paperbound, \$1.75; clothbound, \$3.50.

Reproduction by photo-offset of 19 original papers which have been important in the development of compressible fluid theory. This is followed by a bibliography (pp. 263–282) of papers in compressible fluid theory up to about 1949.

Penney, W. G., and Pike, H. H. M. Shock waves and the propagation of finite pulses in fluids. Reports on Progress in Physics 13, 46–82 (1950).

Expository paper.

Lin, C. C. A new variational principle for isenergetic flows. Quart. Appl. Math. 9, 421–423 (1952).

In steady isoenergetic flow Crocco's stream function $\Psi(x, y)$ is defined by $\Psi_y/u=-\Psi_x/v=y^e(1-w^2)^{1/(\gamma-1)}$, where $e=0(1)$ for plane (axisymmetric) flow, u, v are cartesian (cylindrical) velocity components expressed in units of the maximum attainable speed, and $w^2=u^2+v^2$. The function Ψ satisfies

$$(c^2-u^2)\Psi_{xx}-2uv\Psi_{xy}+(c^2-v^2)\Psi_{yy}-ec^2\Psi_y/y \\ =y^{2e}(1-w^2)^{(\gamma+1)/(\gamma-1)}(w^2-c^2)(\gamma-1)(2\gamma R)^{-1}dS/d\Psi,$$

where c is the local speed of sound and $S(\Psi)$ the entropy per unit mass. It is shown that this is the variational equation for $I=\int\int(p+\rho w^2)y^e dxdy$ with the boundary condition $\Psi_n\delta\Psi=0$, where p is pressure, ρ density.

J. H. Giese.

Lin, C. C., and Shen, S. F. An analytic determination of the flow behind a symmetrical curved shock in a uniform stream. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 2506, 45 pp. (1951).

A method of determining approximately by polynomial solutions the flow pattern behind a detached shock in two-dimensional and axisymmetrical flow is given. As a result of the work, the authors suggest that for axisymmetrical flow at a given Mach number the density variation between shock and nose may be a universal function of distance relative to the total distance between them. This is verified experimentally in the flow past a sphere at $M=1.7$. The authors appear to be unacquainted with the extensive work of Cabannes [Recherche Aéronautique no. 21, 3–7 (1951); these Rev. 13, 180] in similar fields.

M. J. Lighthill.

Ludford, Geoffrey S. S. The classification of one-dimensional flows and the general shock problem of a compressible, viscous, heat-conducting fluid. J. Aeronaut. Sci. 18, 830–834 (1951).

Using the Navier-Stokes equations, and von Mises' [J. Aeronaut. Sci. 17, 551–554 (1950); these Rev. 12, 299] variables and method of solution, the author classifies different types of solutions for one-dimensional flows of a viscous and heat-conducting gas. Except the conventional shock solution, the physical significance of these solutions is open to conjecture.

H. Tsien (Pasadena, Calif.).

des Clers, Bertrand, and Chang, Chieh-Chien. On some special problems in linearized axially symmetric flow. J. Aeronaut. Sci. 18, 127–138 (1951).

Axially symmetric flows along an infinitely long body with sinusoidal corrugations $r_b=r_0+\epsilon\sin\alpha x$ are studied by means of the linearized equation

$$\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}+(1-M^2)\frac{\partial^2\varphi}{\partial x^2}=0,$$

where φ is the disturbance velocity potential and M is the free-stream Mach number. Three types of flow: solid wall at $r=h$, free jet boundary at $r=h$, and free flight ($h\to\infty$), are considered for both subsonic and supersonic speeds. Among other problems are studied the wind tunnel correction, and deviations from the $(1-M^2)^{-\frac{1}{2}}$ Prandtl-Glauert rule in the subsonic case, and the pressure distribution and wave drag of the body in the supersonic case. Finally, as a simplified model of supersonic boundary-layer theory, mixed flows such that $M=M_1<1$ for $r_b<r<r_i$ and $M=M_2>1$ for $r>r_i$ are considered. The shape of gaseous interface (given as $r_i=i+\epsilon_i\sin(\alpha x-\psi_1)$) as well as the pressure distribution and wave drag on the body are investigated, as influenced by the "boundary layer" thickness $T=i-r_0$.

I. Imai.

Schultz-Piszachich, W. Beitrag zur formelmässigen Berechnung der stationären Geschwindigkeitsverteilung umströmter Drehkörper im Unter- und Überschallbereich. Österreich. Ing.-Arch. **5**, 289–303 (1951).

The author has made use of the fact that the equation

$$\beta^2\varphi_{xx}-\varphi_{rr}-\frac{1}{r}\varphi_r=0$$

has a solution of the form

$$\varphi=\sum_{n=0}^{\infty}\frac{1}{(n!)^2}\left(\frac{\beta r}{2}\right)^{2n}\frac{\partial^{2n}}{\partial x^{2n}}\left[a(x)+b(x)\left(k_n+\ln\frac{\beta r}{2(1+x)}\right)\right],$$

where $k_0=0$, $k_n=-\sum_{m=1}^{n}(1/m)$, and $a(x)$, $b(x)$ are arbitrary, to develop a theory of supersonic flow about thin pointed bodies of revolution (about the x-axis). Assuming the profile to be of the form $r=r(x)=\epsilon\rho(x)$, $0<\epsilon\ll1$, and neglecting terms in ϵ above the second, $a(x)$ and $b(x)$ are determined relatively simply in terms of $r(x)$, leading to fairly simple expressions for the components of velocity, etc. A similar procedure is applied to determine the flow about a thin body of revolution having a small angle of yaw. Similar analyses are carried through in the subsonic and in the incompressible cases. *E. Pinney* (Berkeley, Calif.).

Fuller, Franklyn B., and Briggs, Benjamin R. Minimum wave drag of bodies of revolution with a cylindrical center section. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2535**, 34 pp. (1951).

Von Kármán's formula for the wave drag of slender bodies of revolution, which has been shown to be accurate to a high order of approximation, is employed. The investigation is parallel to earlier investigations by several authors of the shape of bodies for minimum wave drag; here the bodies are required to be symmetrical fore-and-aft and to have cylindrical central sections extending over fixed proportions of the total length. Other fixed dimensions are total length, frontal area, and volume. The results reduce to the earlier results when the cylinder is eliminated. The drag penalty due to a cylindrical central part is found to be relatively small. An attempt is made to analyze the minimum-drag problem with skin-friction included. An approximate formula is used for the surface area. This analysis leads to an optimum thickness ratio, whereas of course the wave drag decreases continually with decreasing thickness ratio.

W. R. Sears (Ithaca, N. Y.).

Adams, Mac C. Determination of shapes of boattail bodies of revolution for minimum wave drag. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2550**, 20 pp. (1951).

Lomax, Harvard, and Heaslet, Max. A. Generalized conical-flow fields in supersonic wing theory. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2497**, 45 pp. (1951).

The methods of the authors' previous papers [Tech. Rep. Nat. Adv. Comm. Aeronaut. no. **889** (1947); no. **961** (1950)] are applied to the problem of higher order supersonic conical fields. The load distribution on a lifting surface and the calculation of a symmetrical wing with given pressure distribution are particularly studied, and many examples are given. The second type of problem named is found not always to have a unique solution. *M. J. Lighthill.*

Krasil'ščikova, E. A. The pressure distribution on a lifting surface. Doklady Akad. Nauk SSSR (N.S.) **79**, 747–750 (1951). (Russian)

In linearized steady irrotational supersonic flow the velocity potential for a thin wing can be obtained from a source distribution, with intensity proportional to the local angle of attack, over the wing's projection onto one of the coordinate planes. The linearized pressure distribution can be expressed in terms of double integrals over and line integrals around regions bounded by segments of characteristics and of the boundary of the wing's projection. The double integrals vanish if the wing is plane. Some line integrals vanish if the wing tips' projections are straight and parallel to the undisturbed flow. The rest vanish on the intersection of the wing's projection with the locus of the fourth vertex of characteristic parallelograms, one vertex of which is on the leading edge, while the two adjacent vertices are also on the wing's boundary. Thus there exist wings with regions of negative lift. *J. H. Giese.*

Pistolesi, Enrico. Precisazioni sul metodo delle singolarità nell'aerodinamica supersonica delle ali. Aerotecnica **31**, 86–89 (1951).

Let $\varphi_S=\int_0^{x-\lambda|z|}d\xi\int_{\eta_1}^{\eta_2}f(\xi,\eta)\varphi_{1s}d\eta$ be the velocity potential of a source distribution on the projection of a wing onto $z=0$, where $y=\eta_1$ and $y=\eta_2$ are the intersections of the projection with the Mach forecone of (x, y, z), $\lambda=(M^2-1)^{\frac{1}{2}}$, and the source potential $\varphi_{1s}=\{(x-\xi)^2-\lambda^2[(y-\eta)^2+z^2]\}^{-\frac{1}{2}}$. The author verifies that $(\partial\varphi_S/\partial z)_{z=0}=\mp\pi f(x,y)$ (0) on (off) the wing, and that φ_S satisfies the wave equation

$$\lambda^2\varphi_{xx}-\varphi_{yy}-\varphi_{zz}=0.$$

There are parallel discussions of the potential of a distribution of elementary horseshoe vortices based on $\varphi_{1v}=(x-\xi)z[(y-\eta)^2+z^2]^{-1}\varphi_{1s}$ and of the doublet potential $\varphi_D=\partial\varphi_S/\partial z$. If $f(x,y)$ is independent of y, familiar relations are obtained between $\partial\varphi/\partial x$ and $\partial\varphi/\partial z$ on $z=0$ in plane flow. *J. H. Giese* (Havre de Grace, Md.).

Sears, W. R., and Tan, H. S. The aerodynamics of supersonic biplanes. Quart. Appl. Math. **9**, 67–76 (1951).

The linear theory of the biplane of finite span at supersonic speeds is developed, with special emphasis on the "Busemann biplane" sectional shape. The spanwise distribution of wave drag at the tips is calculated for the particular Mach number at which the sectional drag is zero. The total wave drag coefficient on linear theory is found to be $0.823\delta^2/(M^2-1)A$, where δ is the leading edge angle of the wing sections and A is the aspect ratio of each of the two wings. *M. J. Lighthill* (Manchester).

Mitchell, A. R. Application of relaxation to the rotational field of flow behind a bow shock wave. Quart. J. Mech. Appl. Math. **4**, 371–383 (1951).

The relaxation method of R. V. Southwell is used to calculate a two-dimensional rotational mixed subsonic-supersonic flow field in the neighborhood of a rectangular obstacle. It is assumed that the location of the head shock wave is completely known a priori. (In the example carried out in the text, this information is obtained from a photograph.) The conditions along the shock front combined with the conditions along the contour of the obstacle constitute the boundary values used in the solution. Particular difficulty is encountered in determining the sonic line. This is accomplished by a trial-and-error process. *H. Polachek.*

Hamaker, Frank M., Neice, Stanford E., and Eggers, A. J., Jr. The similarity law for hypersonic flow about slender three-dimensional shapes. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2443**, 22 pp. (1951).

The authors extend the hypersonic similarity laws for slender bodies to three-dimensional shapes with lift. The theory is in agreement with experimental results on pressure at the surface of cones at yaw. *H. Tsien.*

✶Siegel, Keeve M. An analysis of a Stokesian fluid. Engineering Research Institute, Rep. no. **UMM-52**, University of Michigan, Ann Arbor, Mich., 1950. iii+13 pp.

The author tries to dress up the well known classical Stoke's solution of viscous incompressible flows at very small Reynolds numbers for high altitude flights of missiles at slow speeds. He forgets however that at small densities, the boundary condition at the solid surface must be properly modified to take account of the velocity slip. *H. Tsien.*

Lin, T. C. The effect of variable viscosity and thermal conductivity on the high-speed Couette flow of a semi-rarefied gas. University of Washington, Engineering Experiment Station, Bulletin no. **118**, 26–60 (1951).

The author uses the term "flow of semi-rarefied gas" to mean slip flow, i.e., flow where the ratio of mean free path to the characteristic length of the flow lies between 0.01 to 1. The appropriate dynamic equations to use are the so-called Burnett equations, containing additional terms for the viscous stress and for the heat flux, bearing generally the ratio M^2/Re to the corresponding terms in the Navier-Stokes equations (M Mach number, Re Reynolds number). These additional terms are thus of importance only for high speed slip flows. The boundary conditions for such flows were obtained by R. Schamberg [Thesis, California Institute of Technology, Pasadena, California, 1947]. Schamberg has also calculated the Couette flow for this case, assuming, however, constant viscosity coefficient and heat conduction coefficient, both independent of temperature. The present calculation takes account of the variation of these coefficients with temperature. The author also uses the corrected constants for Maxwell molecules and new constants for rigid elastic spherical molecules, both given by Wang Chang and Uhlenbeck. For the boundary conditions, Schamberg's constants are used. Since Schamberg used the uncorrected constants for Maxwell molecules, there may be a slight error here.

The equations for the Couette flow can all be integrated by simple quadrature. The solution is determined by retaining terms up to $(M/\mathrm{Re})^2$, as high order terms are beyond the accuracy of the Burnett equations. For the case of equal wall temperatures the effect of variable viscosity and thermal conductivity tends to increase the slip velocity and the temperature jump, as well as the friction coefficient and the heat transfer, especially at high Mach number and for decreasing density of the gas. Furthermore, at high Mach numbers, the temperature jump plays a more important part in the friction through the increase of viscosity coefficient with temperature and the friction coefficient actually increases with reduction in gas density. *H. Tsien.*

Berson, F. A. On the factors controlling the instability of long waves in zonal currents. Ark. Geofys. **1**, 187–236 (1951).

The motion of the atmosphere is considered on the basis of the linearised equations of motion and of continuity together with the assumption that an additional physical variable, called the polytropic characteristic, has a zero derivative "following the motion". The rotation of the earth is neglected. The wave-solutions of the equations are studied and criteria of instability obtained. As examples of the many results obtained may be cited (i) that horizontally uniform baroclinic zonal currents in middle and high latitudes are dynamically unstable; (ii) that the properties of waves of maximum growth-rate explain features of rapidly deepening disturbances in the westerlies; (iii) small-scale mechanical turbulence is chiefly controlled by the Richardson number whilst large-scale turbulence is controlled by this number together with the dimensions of incipient eddies.

G. C. McVittie (London).

Scorer, R. S. Gravity waves in the atmosphere. Arch. Meteorol. Geophys. Bioklimatol. Ser. A. **4**, 176–193 (1951).

Gravity waves of period short enough for the earth's rotation to be neglected are considered. The waves are propagated through air moving horizontally whose velocity and static stability vary with height. The problem is reduced to a differential equation for the vertical velocity as a function of height. The boundary conditions are zero vertical velocity at the ground and finiteness of kinetic energy. Examples are worked out for airstreams in which the stability and velocity are given functions of the height; it is proved that very different results are obtained according as these functions are changed. Unstable waves are also considered and it is argued that they can occur only in very special circumstances and cannot lead to a phenomenon in which one wavelength predominates. *G. C. McVittie.*

Thrane, P. Some hydrodynamical properties of simple atmospheric oscillations with applications to the semi-diurnal oscillation. Geofys. Publ. Norske Vid.-Akad. Oslo **18**, no. 1, 36 pp. (1951).

The linearized hydrodynamical equations, including the energy equation, are applied to the problem of tidal oscillations of the atmosphere. The gravitational potential is chosen so that the isobaric surfaces are parallel to the earth's surface. The tidal waves are created by a periodic supply and removal of heat. It is shown that in a barotropic model Margules's 24-hour period oscillation cannot be obtained, if the exact hydrodynamical equations are used. For the sake of mathematical simplicity waves near the Pole are considered, which have either 24 or 12 hour periods, and it is shown that the 12-hour waves can develop whereas the conditions are unfavourable for the development of the 24-hour waves. Non-linear perturbation equations are also considered and it is concluded that a slow secular west-east motion at the Equator is superposed on the oscillations. The author then returns to the 12-hourly oscillation and, by numerical integrations, obtains approximate values for the variation of the supplied heat and of the vertical velocity. When both the 12-hour and the 24-hour oscillations are present, it appears probable that the former will take over the greater part of the heat transport.

G. C. McVittie (London).

Thompson, W. B. Thermal convection in a magnetic field. Philos. Mag. (7) **42**, 1417–1432 (1951).

This paper is a more extended version of a paper already published by the author [Proc. Second Canadian Math. Congress, Vancouver, 1949, Univ. of Toronto Press, 1951, pp. 196–205; these Rev. **13**, 398]. The criterion obtained in the earlier paper for the onset of convection in a horizontal

layer of fluid in the presence of a magnetic field and heated below and for the case when both the bounding surfaces are free, is re-derived; however, in this paper the uniqueness of the solution assumed in the earlier paper is established. The principal point in which the present paper is an advance over the earlier one is that the principle of the exchange of stabilities (i.e., the assumption that the situation in marginal stability is characterized by $\partial/\partial t=0$) is examined; for the criterion for stability obtained on this principle will not be valid if instability sets in by oscillations of increasing amplitude. Whether this latter eventuality can arise is therefore examined. By neglecting viscosity and considering the general time-dependent equations for the case when both the bounding surfaces are free, the author shows (i) that the principle of the exchange of stabilities will be valid if $4\pi\kappa\mu\sigma<1$, where μ, κ and σ denote the magnetic permeability, thermometric conductivity and electrical conductivity, respectively, and (ii) that under terrestial conditions the required inequality will be satisfied. [It can be shown that the condition $4\pi\kappa\mu\sigma<1$ for the principle of the exchange of stabilities to be valid, remains true even if viscosity is included; also, under astrophysical conditions this inequality will not be satisfied; see a forthcoming paper by the reviewer.] *S. Chandrasekhar* (Williams Bay, Wis.).

Spence, R. D., and Granger, Sara. The scattering of sound from a prolate spheroid. J. Acoust. Soc. Amer. **23**, 701–706 (1951).

This paper presents the results of calculations of the scattering of a plane sound wave from a prolate spheroid. Scattering patterns are given for the major axis equal to λ/π, $2\lambda/\pi$, $3\lambda/\pi$, for the ratio of axes equal to 0.10, 0.20, 0.29, 0.37, and for the angle of incidence measured from the major axis equal to 0°, 30°, 60°, 90°. *Author's summary.*

Elasticity, Plasticity

Green, A. E., and Shield, R. T. Finite extension and torsion of cylinders. Philos. Trans. Roy. Soc. London. Ser. A. **244**, 47–86 (1951).

The authors give complete solutions to some fundamental problems of finite elastic strain of an isotropic cylinder. They list a bibliography of some 60 items, but these deal either with development of the general theory or with incomplete or restricted attempts on the problems treated by the authors, whose results appear to be entirely new. In the first part of the paper the strain energy is left a perfectly arbitrary function of the strain invariants.

Problem 1. Upon an extension of any amount, a small twist is superposed, only terms of first order in the twist per unit length being retained. A complete solution is obtained in terms of the torsion function as determined from the infinitesimal theory. The authors indicate the remarkable result that, given a particular cross-section which is not a circle or circular annulus, for a certain value of the extension λ the twisting couple will vanish to the order of approximation considered. For an incompressible body this value is given by $\lambda^3=(I_0-S_0)/I_0$, where I_0 is the moment of inertia about the axis of twist and S_0 is proportional to the classical torsional rigidity about the axis of twist. For incompressible materials they also obtain the remarkable and important result that the force F required to produce the extension λ and the torque L required to produce the twist ψ are related by the universal formula

$$\frac{F}{L/\psi}=\frac{(\lambda-\lambda^{-2})A_0}{I_0-(I_0-S_0)\lambda^{-3}},$$

A_0 being the unstrained cross-sectional area. This expression, which is independent of the functional form of the strain energy and which was obtained for the special case of a circular cylinder by Rivlin [same Trans. Ser. A. **242**, 173–195 (1949); these Rev. **11**, 627], furnishes the complete answer to the hitherto unsolved classical problem of the change of effective torsional rigidity incident upon large extension. For compressible cylinders the authors obtain a relation similar in form though dependent upon the nature of the strain energy.

Problem 2. Upon a uniform dilatation of any amount a small twist is superposed. In the case of circular cylinder or tube, the effective torsional rigidity is not altered, but for other cross-sections results analogous to those for Problem 1 are obtained. The authors state some generalizations to the case of small twist superposed upon combined extension and dilatation.

In the second part of the paper the authors confine their attention to a Mooney material, an incompressible material in which the strain energy is exactly a quadratic function of the principal extensions. In such a material there are only two elastic moduli.

Problem 3. In pure torsion of a cylinder of simply-connected cross-section, terms up to and including the second power of the twist are retained in all expressions. The method is semi-inverse, a class of displacements defined in terms of the classical torsion function being assumed. The problem is reduced to that of solving an elaborate second order linear system, subject to complicated boundary conditions. The general solution is obtained by a complex variable method introduced for the infinitesimal theory by Muskhelišvili. The results exhibit the complexity to be expected from the generality of the problem. The authors show how they simplify for certain special cross-sections. As is now well known from general considerations, the effective torsional rigidity is not altered, but a normal force proportional to the square of the angle of twist is required. The authors give a general formula for this force.

Problem 4. A cylinder of simply-connected cross-section is subjected to an extension of any amount, then twisted; second powers of the twist are retained. The authors note that because of the finite extension the Mooney form of strain energy is not sufficient to exhibit all possible second order terms in torsion, and further assumptions restrict their results to the case when the axis of twist is the line of centroids. An additional extension proportional to the square of the twist is included in the class of deformations considered. A general solution is again obtained by complex variable methods. The authors obtain the solution for an elliptic cylinder, but not by specializing their general results. By-products are formulae for second order effects in torsion combined with longitudinal extension proportional to the square of the twist. The authors note that at least in the special case when one Mooney constant vanishes it is possible to adjust the extension so as to annul the normal force on the ends, and that this extension is positive or negative according as the ratio of semi-axes, ordered according to the sense of the twist, is less than or greater than $\sqrt{3}$. This result may be regarded as a first approximation to the solution of the problem of moderate twist of a cylinder with free ends.

Rarely has a single paper contributed so many new and important results to the theory of elasticity.

C. Truesdell (Bloomington, Ind.).

Timpe, A. Spannungsfunktionen achsensymmetrischer Deformation in Zylinderkoordinaten. Z. Angew. Math. Mech. **31**, 220–224 (1951). (German. English, French, and Russian summaries)

Starting with the equations governing dilatation and rotation for an axially-symmetric torsion-free stress-field the stress function given by Love [Mathematical theory of elasticity, 4th ed., Cambridge, 1927, p. 276] is presented. Polynomial solutions up to the sixth degree are given representing circular plate problems. They are combined to give the solution for a uniformly loaded simply supported circular plate to be found in Timoshenko, Theory of elasticity, p. 317 [McGraw-Hill, New York, 1934]. *E. H. Lee.*

Ōkubo, H. On the two-dimensional problem of a semi-infinite elastic body compressed by an elastic plane. Quart. J. Mech. Appl. Math. **4**, 260–270 (1951).

A rectangular elastic block $-2h\leqq y\leqq 0$, $|x|\leqq 1$ is pressed against a smooth semi-infinite elastic body $y\geqq 0$ by uniform pressure on the face $y=-2h$, h assumed large. The stresses are determined by considering a Fourier series expansion for the interface pressure and writing an approximate solution for the block which includes non-zero σ_x on the faces $|x|=1$. In an appendix this shortcoming is shown to have negligible influence away from the contact eges. An equivalent power series expansion for the interface pressure gives the solution in the body $y\geqq 0$. Compatible interface displacement gives equations for the coefficients. The solution depends on the ratios of Young's moduli and five values are discussed including 0, 1 and ∞. Stress concentration occurs at the contact edges of the block. Convergence seems good away from these edges. The problem is considered as one of plane stress, but, in view of the stress concentration, modified elastic coefficients for plane strain would be physically more acceptable. *E. H. Lee.*

Birman, S. E. On an effective variant of a solution of a problem of the theory of elasticity for an infinite strip. Akad. Nauk SSSR. Prikl. Mat. Meh. **14**, 665–669 (1950). (Russian)

In a previous paper [Doklady Akad. Nauk SSSR (N.S.) **62**, 187–190 (1948); these Rev. **10**, 341] the author has given the solution of the problem of determining the stresses occurring in an infinite plane strip, $-\infty\leqq x\leqq +\infty$, $-a\leqq y\leqq a$, in elastic equilibrium in terms of two analytic functions which are given explicitly by certain infinite integrals involving the applied boundary load. In the present paper the author applies this solution to the particular case of compression of the infinite strip by oppositely directed concentrated forces of equal magnitude, applied at the points $(0, a)$ and $(0, -a)$, and obtains the solution obtained in a different way by L. N. G. Filon [Philos. Trans. Roy. Soc. London. Ser. A. **201**, 63–155 (1903)] and E. Melan [Beton und Eisen **18**, 83–85 (1919)]. The numerical results given by these authors were arrived at by laborious computations, due to the discontinuity of the boundary load in the infinite integral. The variant referred to in the title of this paper consists in rewriting the solution as the sum of two terms, the first of which is essentially the well known Schwarz integral for a half-plane. This device of rewriting the solution leads to a simplification of the computations for an approximate solution, which is compared with the numerical solutions of Filon and Melan. The case of a boundary load uniformly distributed over $-b\leqq x\leqq b$ is also considered.

J. B. Dias (College Park, Md.).

Sengupta, A. M. Some problems of elastic plates containing circular holes. I. Bull. Calcutta Math. Soc. **43**, 27–36 (1951).

Using the method of Jefferey [Philos. Trans. Roy. Soc. London. Ser. A. **221**, 265–293 (1920)] the author finds solutions for the Airy stress function χ in terms of bi-polar coordinates for an infinite plate with two unstressed equal circular holes, subject to the action of (I) a couple of moment M and (II) a centre of pressure midway between them. Thence he determines the stresses round the edges of the holes as rapidly convergent infinite series, and gives tables of numerical values of these stresses round the holes in each case for a particular value of the ratio of the distance between the centres of the holes to the diameter of a hole.

R. M. Morris (Cardiff).

Morgan, A. J. A. Uniformly loaded semi-infinite wedge-shaped plates. J. Aeronaut. Sci. **18**, 845–847 (1951).

Making the substitution $\omega=x^4f(\eta)$, $\eta=y/x$ in the usual partial differential equation for plate bending $\nabla^4\omega=P(x, y)/D$, the resulting ordinary differential equation yields the general solution

$$f(\eta)=C_1\eta+C_2(\eta^2-\tfrac{1}{3})+C_3\eta^3+C_4(\eta^4-3\eta^2)+\frac{1}{8}\frac{P}{D}\eta^2$$

if P is constant. The author shows that the above solves the problem of the title for various boundary conditions on the two edges, and gives details for the case where one edge is clamped and the other free. *H. D. Conway.*

Fridman, M. M. The bending of a circular plate by concentrated forces. Akad. Nauk SSSR. Prikl. Mat. Meh. **15**, 258–260 (1951). (Russian)

The author considers the problem of the bending of a thin circular isotropic plate of unit radius (represented by the unit circle $|z|\leqq 1$, $z=x+iy$) when acted upon by $M+N$ concentrated forces P_k ($k=1, \cdots, M, M+1, \cdots, M+N$) at the M interior points

$$z_k=r_ke^{i\theta_k}, \quad |z_k|<1, \quad k=1, \cdots, M,$$

and the N boundary points

$$z_k=r_ke^{i\theta_k}, \quad |z_k|=1, \quad k=M+1, \cdots, M+N,$$

where

$$\sum_{k=1}^{M+N} P_k=0, \quad \sum_{k=1}^{M+N} z_kP_k=0.$$

The deflection, bending and twisting moments, and shearing stresses can all be given in terms of two analytic functions $\varphi(z)$ and $\chi(z)$ [Lehnickiĭ, same journal **2**, 181–210 (1938)]. The author first determines the functions $\varphi(z)$ and $\chi(z)$ (using the procedure in Mushelišvili [Some fundamental problems in the mathematical theory of elasticity, Moscow-Leningrad, 1935, §69]) and then gives explicit formulas for the deflection, bending and twisting moments, and shearing stresses in terms of these two analytic functions. As special cases of his formulas the author finds in closed form the solution of certain particular cases of the present problem, which were considered earlier by Lur'e [same journal **4**, 93–102 (1940)] and Nádai [Die elastischen Platten . . . , Springer, Berlin, 1925, p. 193; Physikalische Z. **23**, 366–376 (1922)]. *J. B. Dias* (College Park, Md.).

Craemer, H. Einige Iterations- und Relaxationsverfahren für drehsymmetrisch beanspruchte Zylinderschalen. Österreich. Ing.-Arch. **6**, 35–42 (1951).

Federhofer, K. Berechnung des kreiszylindrischen Flüssigkeitsbehälters mit quadratisch veränderlicher Wandstärke. Österreich. Ing.-Arch. **6**, 43–64 (1951).

The particular case of a circular tank with wall thickness varying as the square of the axial distance from any reference point is shown to lead to a solution in elementary instead of hypergeometric functions. Numerical values for moment shear deflection and circumferential stress are computed for tanks free at the top and fixed at the bottom with a ratio of wall thickness of 1 to 2. Comparison is made with a minimum potential energy power series solution by Pöschl.

D. C. Drucker (Providence, R. I.).

Yu, T. M. Shearing stresses in curved beams. Acad. Sinica Science Record **2**, 401–408 (1949).

The author considers the pure bending of a curved beam with a uniform cross section and a circular axis, the cross-section being symmetrical with respect to the plane of the axis. When the depth of the cross-section is much smaller than the radius of the central axis, the same variation of stress as for straight beams is assumed. When the depth is comparable with the radius of the central axis, E. Winkler [Civilingenieur (Neue Folge) **3**, 232–246 (1857)] assumed that cross-sections remain plane and hence deduced that the normal stresses over the cross section follow a hyperbolic law. This law has been verified in the cases when exact solutions exist [S. Timoshenko, Theory of Elasticity, McGraw-Hill, New York, 1934, p. 61]. In the present paper the author starts with the Winkler theory, and writes down the macroscopic equations of equilibrium of the beam to deduce an expression for the shearing stress acting on the cross-section. Numerical results are presented for cases when the cross-section is a rectangle and a circle. These results agree well with those obtained from known exact solutions. Note: the same problem has been solved in the same way by R. Kappus [Recherche Aéronautique no. **21**, 49–51 (1951)].

G. E. Hay (Ann Arbor, Mich.).

Mitra, D. N. Torsion and flexure of a beam whose cross-section is a sector of a curve. Bull. Calcutta Math. Soc. **43**, 41–45 (1951).

In earlier papers [same Bull. **41**, 28–40, 125–128, 153–158 (1949); **42**, 188 (1950); these Rev. **11**, 68, 289, 287; **12**, 456] the author has applied the method of conformal mapping and Schwarz's principle of reflection of S. Ghosh [ibid. **39**, 1–14 (1947); these Rev. **9**, 256] to solve torsion and flexure problems. In the present paper two successive conformal mappings are employed to map the sector on the unit half-circle. For the torsion problem an analytic function is then constructed so that its imaginary part vanishes on the real axis corresponding to the two straight lines in the original sector boundary. By analytical continuation the function is extended and properly defined in the lower half unit circle. Similar methods apply to the flexure problem. Some special values of the sector angle require special treatment. The analysis is for a general type of bounding curve; however it furnishes known results for sectors of a circle.

D. L. Holl (Ames, Iowa).

Hopkins, H. G., and Brown, E. H. The effect of internal pressure on the initial buckling of thin-walled circular cylinders under torsion. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2423** **(9538)**, 13 pp. (1951).

The aerodynamic suction experienced in flight by the engine nacelle of a jet propelled aircraft is equivalent to an internal pressure on the nacelle skin thus affecting its stability. It is found that the effect of this internal pressure on a circular cylinder under torsion is to increase the critical shear stress, to decrease the circumferential wave length, and to increase the angle between the axis of the cylinder and the direction of the waves. Agreement between theoretical and experimental values of the critical shear is good. The analysis shows that the displacements (u, v, w) axially, circumferentially and radially satisfy a system of partial differential equations of the eighth order. Boundary conditions at the ends of the cylinders where the ring stabilizers are attached have little effect on the stability under clamped or simply supported edge conditions. There is a striking increase in torsional stability by the addition of internal pressure.

D. L. Holl (Ames, Iowa).

Funk, P. Über ein Stabilitätsproblem bei den durch Krümmung steif gemachten Messbändern. Österreich. Ing.-Arch. **5**, 387–397 (1951).

The discussion is limited to bending in one plane without twisting of a tape with a circular arc cross-section. Buckling is sudden from the arched configuration to the locally flattened tape and is treated as a discontinuous process. A general discussion is given of several extremal problems in analytical and geometrical terms and the difference between neutral instability and the jump type is explained. Diagrams are drawn and numerical values given for strain energy and bending moment against curvature and also for critical points.

D. C. Drucker (Providence, R. I.).

Kappus, R. Le "schéma du champ homogène" rectangulaire ou oblique. Recherche Aéronautique no. **23**, 51–60 (1951).

Falkenheiner, H. La systématisation du calcul hyperstatique d'après l'hypothèse du "schéma du champ homogène." Recherche Aéronautique no. **23**, 61–65 (1951).

A simple approximate scheme for analyzing elastic structures composed of stiffened rectangular panels is based on the possibility of considering the sheet of the panel as a "shear field" which transmits only constant shearing forces. The first paper presents a generalization of this scheme in which the sheet is also supposed to transmit forces parallel to the edge stiffeners which are constant in each panel. It is pointed out that the generalization is essential for oblique panels, and is also of particular importance for rectangular panels in which the sheet and stiffener sections are of comparable order of magnitude. A numerical example is included. The second paper presents a systematic matrix formulation of the relevant technique.

F. B. Hildebrand (Cambridge, Mass.).

Kuhelj, Anton. On the determination of internal forces in two-spar wings. Acta Tech. **1**, 13 pp. (1950). (Slovenian. English summary)

Pólya, G. A note on the principal frequency of a triangular membrane. Quart. Appl. Math. 8, 386 (1951).

The principal frequency of a membrane of triangular shape is exactly known for the 45°-45°-90° and the 60°-60°-60° triangles [Rayleigh, The theory of sound, v. 1, Macmillan, London, 1877, §199]. In this note an exact solution of comparable simplicity is obtained for the 30°-60°-90° triangle. However, the general expression for the solution in this last case was already given by B. R. Seth [Proc. Indian Acad. Sci., Sect. A. 12, 487–490 (1940); these Rev. 3, 123].

J. B. Diaz (College Park, Md.).

Zaicev, L. P., and Zvolinskiĭ, N. V. Investigation of the axisymmetric head wave arising on the plane boundary dividing two elastic liquids. Izvestiya Akad. Nauk SSSR. Ser. Geofiz. 1951, no. 5, 40–50 (1951). (Russian)

The authors make use of the method of Smirnov and Sobolev [Akad. Nauk SSSR. Trudy Seismol. Inst. no. 20 (1932)] and of some results of a previous paper [Izvestiya Akad. Nauk SSSR. Ser. Geograf. Geofiz. 15, no. 1, 20–39 (1951); these Rev. 12, 650]. A class of solutions of a wave equation is represented by the real part of an arbitrary function of a complex variable. This variable is determined by a linear function in time and in rectangular coordinates (x, z) in a plane. A two-dimensional problem can be used in order to obtain a solution of an axisymmetric problem. This is done by the substitution $x=r\cos\omega, z$, where r, ω, z are cylindrical coordinates. The method is applied to the axisymmetric head wave arising on the plane boundary between two liquids when the direct wave is emitted by a point source producing a disturbance proportional to time (impulse of short duration). *W. S. Jardetsky.*

Yosiyama, Ryoiti. Elastic waves from a point in an isotropic heterogeneous sphere. III. Bull. Earthquake Res. Inst. Tokyo 19, 185–205 (1941). (English. Japanese summary)

[For parts I and II see same Bull. 11, 1–13 (1933); 18, 41–56 (1940); these Rev. 2, 32.] The heterogeneity is assumed small, and first order effects due to the gradients of the density and elastic constants are considered. The problem reduces to the solution of $\varphi_{tt}=v^2\Delta\varphi-\alpha^2\varphi$, where α is a function of the gradients of the density and elastic constants. Steady oscillatory solutions are sought. In general longitudinal waves and two types of transverse waves occur. One of the types of transverse waves and the longitudinal waves interact, so that volume change develops for an initially transverse motion and vice-versa. Dispersion effects occur, two stress peaks may be developed, and also an oscillatory tail. These are discussed in general and also for specific examples. The complexity demands the use of approximate analysis. Near a source the behaviour approaches that for a homogeneous medium, and the dissipative effects decrease with decreasing period. Seismic waves are discussed to attempt prediction of the mode of earthquake initiation. Lack of coupling between P and S waves observed suggests small heterogeneity in the body of the earth.

E. H. Lee (Providence, R. I.).

Caloi, Pietro. Teoria delle onde di Rayleigh in mezzi elastici e firmo-elastici, esposta con le omografie vettoriali. Arch. Meteorol. Geophys. Bioklimatol. Ser. A. 4, 413–435 (1951).

L'auteur retrouve à l'aide du calcul vectoriel, pris sous la forme de Burali-Forti et Marcolongo, les équations de l'élasticité des corps isotropes et la théorie classique des ondes de Rayleigh, soit en milieu élastique, soit en milieu à frottement intérieur de Voigt. Résultats numériques déjà publiés [Publ. Bureau Central Seismol. Internat. Sér. A. Trav. Sci. 17, 89–108 (1950); ces Rev. 12, 772].

J. Coulomb (Paris).

*__Haskell, Norman A. The dispersion of surface waves on multi-layered media.__ Geophysical Research Papers, no. 9, Air Force Cambridge Research Center, Cambridge, Mass., 1951. 28 pp.

Il est pénible de calculer les courbes de dispersion des ondes de Rayleigh et de Love dans un milieu stratifié dès que le nombre des couches dépasse deux. Pour simplifier ce calcul, W. T. Thomson [J. Appl. Phys. 21, 89–93 (1950); ces Rev. 11, 702] lui a donné une forme matricielle. Sa théorie est ici reprise, corrigée d'une erreur, puis appliquée au calcul de la vitesse des ondes de Rayleigh sous les continents. La comparaison aux observations ne permet pas d'écarter l'hypothèse de Gutenberg suivant laquelle il existerait une couche profonde où les vitesses seraient plus faibles que les vitesses superficielles. *J. Coulomb.*

*__Prager, William. On the boundary value problems of the mathematical theory of plasticity.__ Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 297–303. Amer. Math. Soc., Providence, R. I., 1952.

The progress of plastic deformation for an ideally plastic material is described from its first beginning through the stage of contained plastic deformation to impending plastic flow. The plane strain problem of the tensile specimen with semicircular grooves considered numerically by Allen and Southwell [Philos. Trans. Roy. Soc. London. Ser. A. 242, 379–414 (1950); these Rev. 11, 703] is discussed from the limit design point of view. A simple lower bound is found which proves that the force which first produces a bridge of plastic deformation all the way across the specimen is below the load required for impending flow (collapse).

D. C. Drucker (Providence, R. I.).

Fastov, N. S. On the thermodynamics of plastic deformation. Doklady Akad. Nauk SSSR (N.S.) 78, 251–254 (1951). (Russian)

The expression for the free energy of a plastic body in static equilibrium is generalized to take account of deformation at a finite strain rate; hence, expressions are derived giving the dependence of stress in the plastic region on strain and strain rate, as well as the dependence of the yield stress and maximum elastic strain on the strain rate.

H. I. Ansoff (Santa Monica, Calif.).

Winzer, Alice. Solution to the rolling problem for a strain-hardening material by the method of discontinuities. J. Appl. Mech. 18, 90–94 (1951).

The problem of rolling of a thin wide sheet is solved by a method of discontinuities which approximates a continuous field of stress by constant stress regions with discontinuities along the boundaries between them. The rolls are rigid and the sheet is assumed to be in plane strain state and to exhibit strain hardening. The stress and velocity in adjacent regions depend, in part, on the ratio of the hardening constants (corresponding to the yield constant for perfectly plastic material) in these regions. As a result, for a given stress-strain curve several iterations of the solution may be required before a solution is found which simul-

taneously satisfies the stress-strain equations, the equilibrium equations and the boundary conditions.
H. I. Ansoff (Santa Monica, Calif.).

Eshelby, J. D., and Stroh, A. N. Dislocations in thin plates. Philos. Mag. (7) **42**, 1401–1405 (1951).

The purpose of this paper is to examine several important special cases of dislocations in thin plates, and to contrast their behavior with that of dislocations in semi-infinite bodies. Exact solutions are obtained for the following problems, which are all concerned with rectilinear screw dislocations of Volterra type: (a) the dislocation plane intersects the surface of a semi-infinite body normally; (b) the axis of the dislocation runs normally through an infinite plate; (c) the axis coincides with that of a circular disc. A purely qualitative discussion of (c) for the case of an edge dislocation is also presented. One of the surprising results of this study is the fact that two screw dislocations in a plate attract or repel one another with a short range force, in sharp contrast to the corresponding situation in infinite or semi-infinite bodies. *A. W. Sáenz* (Bloomington, Ind.).

Nabarro, F. R. N. The synthesis of elastic dislocation fields. Philos. Mag. (7) **42**, 1224–1231 (1951).

The main objective of this paper is to study the synthesis of various important static and dynamic dislocations by superposition of suitable elementary solutions.

In §2 it is shown that the static displacement field of an infinitesimal dislocation loop lying in its glide plane, which follows from the work of Burgers [Nederl. Akad. Wetensch., Proc. **42**, 293–325 (1939)], is obtainable by superposition of fields belonging to "the first type of simple solutions" [A. E. H. Love, Mathematical theory of elasticity, Cambridge, 1892, §§131, 132]. In §4, various cases of edge screw dislocations are synthesized by superposition of the elementary loop solutions in §2. The author devotes §5 to studying the field associated with the sudden creation of a dislocation loop, and reduces this problem to that of the sudden application of a force at a point [G. G. Stokes, Trans. Cambridge Philos. Soc. **9**, 1–62 (1851); A. E. H. Love, Proc. London Math. Soc. (2) **1**, 291–344 (1904); Mathematical theory of elasticity, §212]. These results immediately furnish him with the means of investigating a dislocation moving in its glide plane with non-uniform velocity, smaller than the transverse sound velocity, because one may regard such motion as "the result of continuous addition of infinitesimal loops to its boundary". This powerful idea is illustrated in ¶6 by a complete solution of the problem of a screw dislocation moving along a straight line with arbitrary non-uniform velocity smaller than the transferse sound velocity. The simpler problem for the case of uniform velocity had been previously solved by F. C. Frank [Proc. Phys. Soc. Sect. A. **62**, 202–203 (1949)] and J. D. Eshelby [ibid. **62**, 307–314 (1949)].

The reviewer believes that this paper contains ideas and methods which will prove very fruitful in obtaining a complete theory of moving dislocation.
A. W. Sáenz (Bloomington, Ind.).

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Picht, J. Bestimmung eines aus einem (beliebigen) Paraboloidspiegel und einem Zwei-Spiegel-Zusatzsystem bestehenden Drei-Spiegel-Systems, für das die Aufhebung der sphärischen Aberration sowie die Sinusbedingung (Komafreiheit) streng erfüllt ist. Untersuchungen über den Ersatz jener (deformierten) Hilfsspiegel durch einfacher herstellbare Spiegelflächen. Optik **8**, 129–144, 145–161, 193–205, 241–250, 318–334, 337–353 (1951).

It is known that a two-mirror system can be designed entirely free of spherical aberration and coma. But this involves construction of one large surface whose equation is of higher than second degree. The author therefore considers the problem of designing a three-mirror system consisting of a large paraboloidal mirror and two small auxiliary mirrors, the system to be strictly corrected for spherical aberration and coma. With the paraboloidal mirror taken arbitrarily, differential equations are derived for the forms of the auxiliary mirrors and thereby the actual equations of the surfaces are found. The possibility is considered of approximating the auxiliary mirrors by ones which are more easily constructed, i.e., spherical or other second-order surfaces of revolution. Since several parameters can be chosen arbitrarily in the equations of the two auxiliary mirrors, conditions are derived for determining these parameters so as to reduce the other image errors such as astigmatism and curvature of field. The methods are illustrated in detailed numerical examples.
E. W. Marchand (Rochester, N. Y.).

Lefaivre, Jean. A new approach in the analytical study of the spherical aberrations of any order. J. Opt. Soc. Amer. **41**, 647 (1951).

If a plane wave is incident on a circular aperture of radius, a, and there is introduced in the plane of the aperture a phase change proportional to an even power m of the variable radius p, then Kirchoff's equation gives a double integral formula for finding the complex amplitude of the wave reaching a point P beyond the aperture. This integral may be evaluated by expanding the integrand in series, but the author suggests instead that one integration first be performed, obtaining an integrand which is an exponential times a Bessel function, and then expanding either the exponential or the Bessel function in a series to evaluate the integral. *E. W. Marchand* (Rochester, N. Y.).

Morais, C. Le aberrazioni del settimo ordine. Loro numero, forma e studio comparativo colle aberrazioni del 3 e 5 ordine. Ottica (N.S.) **5**, 63–77 (1951).

For the special case of plane parallel plates expressions are obtained for the 7th order aberrations and a trigonometric method is described for the calculation of their coefficients.
E. W. Marchand (Rochester, N. Y.).

Slevogt, H. Zur Auswertung Seidelscher Rechnungen: Vergleich mit Fehlerdarstellungen nach v. Rohr und mit reduzierten Aberrationen. Optik **8**, 537–542 (1951).

This paper contains practical suggestions for comparing the result of the computation of Seidel aberrations with the results of finite ray tracing. The paper answers many questions which the practical designer wants to know.
M. Herzberger (Rochester, N. Y.).

Pidduck, F. B. Electrical notes. XIII. Diffraction of light by a semi-transparent sheet. Quart. J. Math., Oxford Ser. (2) 2, 316–320 (1951).

Fok, V. A. Fresnel diffraction by convex bodies. Uspehi Fiz. Nauk **43**, 587–599 (1951). (Russian)

Radiation from a point source is diffracted by a sphere whose radius is large compared to the wave-length. The question is the nature of the field at large distances from the source in the transitional zone near the geometrical shadow. Using formulae from an earlier work [Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **19**, 916–929 (1949); these Rev. **11**, 563] he represents the field as the sum of two terms, one corresponding to Fresnel diffraction and the other a slowly-varying "background" term, of which only the latter depends on the electrical properties of the sphere. He verifies that his formulae for the transitional zone match with previously known formulae for the illuminated and shadow zones. He thinks it very likely that a similar picture holds for diffraction by a general convex body; cf. his previous papers [Bull. Acad. Sci. URSS. Sér. Phys. [Izvestia Akad. Nauk SSSR] **10**, 171–186 (1946); Acad. Sci. USSR. J. Phys. **10**, 399–409 (1946); these Rev. **8**, 300, 363].

F. V. Atkinson (Ibadan).

Müller, Claus. Über die Beugung elektromagnetischer Schwingungen an endlichen homogenen Körpern. Math. Ann. **123**, 345–378 (1951).

Consider an infinite three-dimensional region A surrounding a finite smooth closed surface F with the volume V enclosed by F having the finite constitutive parameters ϵ_i, μ_i. Let ϵ_a and μ_a be the finite parameters in A. The author proves first that, given an electromagnetic field generated in a region B (B disjoint from V), there exists a solution of Maxwell's equations in the regions A and V. This is done by reducing the problem to a set of Fredholm integral equations as follows: First, a set of electric and magnetic currents are postulated upon F which have the property that they generate the correct reflected field in A but would yield nothing in V if V had the parameters ϵ_a, μ_a. (This corresponds to the usual replacement by dipole layers in potential theory.) Similarly, a second set of layers is postulated on F to replace the sources of the field in so far as the interior of V is concerned. As the boundary F is then approached from the two sides, continuity requirements yield relationships between the two sets of layers and incident field. This allows the elimination of one set of unknowns and provides a pair of vector integral equations in the other set. The kernel is shown to be such as to give a completely continuous operator, and it is then shown that there are no eigensolutions of the pair of integral equations. Finally, the result is checked and shown to satisfy the problem. The author next considers the necessary modifications when V and B are not disjoint.

In many ways the author's third result is the most interesting. In this he returns to the first problem and permits $\epsilon_i \to \infty$ to achieve the boundary condition required for the perfect conductor. The corresponding integral equation formulation makes use of only a magnetic current and it is immediate that there are no eigensolutions if $k=\omega[\epsilon_a\mu_a]^{\frac{1}{2}}$ is not real. [See also F. H. Murray, Proc. Nat. Acad. Sci. U. S. A. **16**, 353–357 (1930).] The surprising part of the formulation is that it permits the author to treat the case k real and show easily that the eigensolutions of the adjoint equation and the incident field meet the necessary orthogonality conditions for the Fredholm alternative.

[Reviewer's note: This last result permits an immediate extension to the problem of an assigned boundary condition for the exterior region. With the solution of the problem of the total reflection of a source known there is required only a judicious choice of three sources in order to construct the three vectors of a dyadic Green's function. This was noted explicitly by J. Schwinger (unpublished notes), and others and is contained inherently in several papers of H. Weyl. Moreover, it appears possible to use such Green's functions to prove the first results of the author's paper without the somewhat unnatural assignment of both magnetic and electric currents at every step. There is, however, not much to recommend such a change from the point of view of the Fredholm theory where n equations are no more trouble than one.] *W. K. Saunders* (Washington, D. C.).

Di Jorio, M. Nuove ricerche sulle leggi fondamentali della propagazione delle onde. Ottica (N.S.) **4**, 58–68, 97–121 (1950); **5**, 9–40 (1951).

This paper discusses at great length the laws of monochromatic wave propagation on the basis of Kirchhoff's integral solution of the equation $(\nabla^2+k^2)F=0$. There appears to be little new in it, despite the title. There are practically no references to the extensive literature of the subject. It is rather surprising to observe that "la formula del Luneberg" of §14 (with a reference to Luneberg's 1944 lectures [Mathematical theory of optics, Brown Univ., Providence, 1944; these Rev. **6**, 107]) was in fact well-known to Rayleigh when he wrote his "Theory of Sound" [Macmillan, London, 1877, 1878] nearly seventy-five years ago. *E. T. Copson* (St. Andrews).

Spence, R. D., and Wells, C. P. Vector wave functions. Comm. Pure Appl. Math. **4**, 95–104 (1951).

The authors review the problem of finding solenoidal solutions of the vector wave equation. The emphasis is on electromagnetic diffraction problems in spheroidal coordinates. *C. J. Bouwkamp* (Eindhoven).

Marcuvitz, Nathan. Field representations in spherically stratified regions. Comm. Pure Appl. Math. **4**, 263–315 (1951).

L'autore affronta il caso più generale del campo elettromagnetico generato in un mezzo a strati sferici concentrici da una qualsiasi distribuzione di sorgenti. Non è possibile analizzare in dettaglio questo ampio e completo studio. La tecnica impiegata è quella dei modi caratteristici o delle onde guidate. Vengono studiate rappresentazioni mediante onde guidate lungo θ o lungo r. Naturalmente in questo modo il problema viene ricondotto a quello delle normali linee di trasmissione e alle relative equazioni. L'autore, dopo avere esposto un procedimento generale per il calcolo delle autofunzioni e degli autovalori, lo applica a un gran numero di casi particolari importanti. *G. Toraldo di Francia.*

Schelkunoff, S. A. Remarks concerning wave propagation in stratified media. Comm. Pure Appl. Math. **4**, 117–128 (1951).

The author presents a general discussion of electromagnetic wave propagation through a horizontally stratified medium. Difficulties in the concept of reflection coefficient for infinite layers are stressed. Cases of zero-reflection, first-order reflection, and strong reflection are treated. A Riccati differential equation is derived for the reflection coefficient. *C. J. Bouwkamp* (Eindhoven).

Saha, M. N., Banerjea, B. K., and Guha, U. C. Vertical propagation of electromagnetic waves in the ionosphere. Proc. Nat. Inst. Sci. India 17, 205–226 (1951).

"We discuss here the exact equations from a standpoint which is likely to throw light on the nature . . . of wave propagation at high geomagnetic latitudes".

C. J. Bouwkamp (Eindhoven).

Försterling, Karl. Wellenausbreitung in der Ionosphäre bei Berücksichtigung des Erdmagnetfeldes bei schiefer Inzidenz. II. Arch. Elektr. Übertragung 5, 209–215 (1951).

This is a continuation of an earlier paper [Arch. Elektr. Übertragung 3, 115–120 (1949); these Rev. 11, 631] on the electromagnetic field in the ionosphere, and consists mainly of more detailed analysis of the field components for nearly perpendicular incidence. The author first neglects the earth's magnetic field and assumes a linear variation of dielectric constant, $\epsilon(z)=\epsilon_1 z-i\epsilon_0=\epsilon_1 z'$. Then if $q^2=\sin^2\phi_0$ (ϕ_0=angle of incidence) is sufficiently near unity, it is possible to define two points $|z'|=\bar{z}$ such that, for $|z'|>\bar{z}$, $1/z'^2\ll q^2$, while for $|z'|<\bar{z}$, $\epsilon_1 z'\ll q^2$. Solutions for the two regions can then be obtained by neglecting different terms in the differential equation, and it is possible to match the two solutions at $z'=\pm\bar{z}$. The analysis for a similar type of layer in the presence of the earth's magnetic field is then outlined, but the notation used is too complicated to permit a brief summary of the results. In general terms it may be said that the case of total reflection in the ionospheric layer is discussed, and also the first approximation obtained by neglecting the coupling between the four component waves.

M. C. Gray (Murray Hill, N. J.).

Kornhauser, E. T. Radiation field of helical antennas with sinusoidal current. J. Appl. Phys. 22, 887–891 (1951).

The author considers a circular helix of radius a and helix angle α with its axis oriented along the z-axis, and determines the distant field on the assumption that the current may be represented approximately by a progressive distribution, $I_0\exp(-i\beta\varphi)$, where $\varphi=z/a\tan\alpha$. This assumption has not been justified theoretically, but is based on experimental results of J. D. Kraus [see, in particular, Proc. I.R.E. 37, 263–272 (1949)], who has shown that over a wide frequency range the velocity ratio $p=v/c$ varies with frequency in such a way that the phase difference between turns is almost precisely the same as between elements of an end-fire array designed for maximum directivity. The phase constant β is written as $\omega a\sec\alpha/cp$ and values of p are chosen to approximate the experimental results as closely as possible. Then the field components can be expressed in terms of certain integrals of the type

$$I=\int_{-\varphi_0}^{\varphi_0}\exp(ix\cos\varphi+i\nu\varphi)d\varphi,\quad x=\frac{\omega a}{c}\sin\theta,$$

$$\nu=\frac{\omega a}{c}\tan\alpha\cos\theta-\beta.$$

This integral can be expanded in series of Bessel functions, and it is shown that only the terms in $J_0(x)$ are important when the helix has several turns, not necessarily an integral number. Comparison with Kraus's experimental results for a 7-turn 12° helix with $\omega a/c=1.06$ and $p=0.83$ shows very close agreement, but for a single turn helix the results differ considerably from those of a square-turn approximation suggested by Kraus. *M. C. Gray* (Murray Hill, N. J.).

Karr, Philip R. Radiation properties of spherical antennas as a function of the location of the driving force. J. Research Nat. Bur. Standards 46, 422–436 (1951).

The author discusses forced oscillations on a perfectly conducting sphere when it is assumed that a driving voltage acts over a narrow strip round a circle of colatitude $\theta=\theta_0$. This is the asymmetric extension of the symmetric problem, $\theta_0=90°$, already analysed by Stratton and Chu [J. Appl. Phys. 12, 236–240 (1941)] and essentially the same method is used. The variations in the radiation field for different values of the radius of the sphere and different locations of the feed-point θ_0 are analyzed in detail and radiation patterns drawn. For very small spheres the location of the feed-point merely changes the magnitude of the radiation field for a given applied voltage; for spheres of radii comparable to the wavelength the single lobe pattern in the symmetric case is tilted away from the feed-point; but as the radius becomes larger the multilobed patterns show no very well-defined characteristics. For the end-fed antenna, $\theta_0=0$, there is actually no radiation (with a finite applied voltage) but as $\theta_0\to0$ the radiation pattern has a finite limiting form, which shows qualitative agreement with experimental measurements on end-fed nonspherical antennas.

The currents and admittances of the various modes of oscillation are also discussed. As usual, the input admittance may be represented as the sum of an infinite number of partial admittances; the series for the conductance converges rapidly, but the susceptance series diverges when a delta-function type of excitation is used. For a finite gap both series converge, but the conductance is not appreciably affected by the width of the gap. Some curves are included showing the variation of the conductance as a function of θ_0 for various values of the radius of the sphere.

M. C. Gray (Murray Hill, N. J.).

Storer, James E. The impedance of an antenna over a large circular screen. J. Appl. Phys. 22, 1058–1066 (1951).

The author considers the problem of an antenna of height h erected vertically in the center of a perfectly conducting circular screen of diameter d. This problem has been solved by Leitner and Spence [J. Appl. Phys. 21, 1001–1006 (1950); these Rev. 12, 462; see also these Rev. 12, 146] for a quarter-wave antenna. Their solution, however, involves an infinite series of spheroidal functions which converges very slowly for large values of d, the region of practical importance in antenna experiments. Assuming that the screen is large, $kd\gg1$ (also $d^2\gg h^2$), the distant field may be considered as the sum of two fields, $H_\varphi=H_\varphi^\infty+\bar{H}_\varphi$, where H_φ^∞ is the known field for an infinite screen and $\bar{H}_\varphi$ the correction for finite d.

In the plane of the screen, $z=0$, the radial component E_r must vanish on the screen, and it will have a value $\xi(r)$ outside the screen. Since $\bar{H}_\varphi$ can be expressed in terms of integrals involving $\xi(r)$ and a Green's function $G(z,r,r')$ the condition that H_φ must be continuous in the plane of the screen, outside the screen, leads to an integral equation for the function $\xi(r)$. This is in turn replaced by a variational formulation, in which a new parameter μ is stationary with respect to variations in ξ. The input impedance is determined by the usual Poynting vector method, and a formula for the difference between the impedance for a finite and for an infinite screen is obtained. Assuming an approximate value for $\xi(r)$ suggested by Sommerfeld's solution of the problem of the diffraction of a plane wave by a semi-infinite

screen, the final expression for the impedance difference is

$$Z-Z_0=\frac{\zeta_0}{2\pi kdi}e^{ikd}\left|k\int_0^h \frac{I(z)}{I(0)}dz\right|^2,$$

where $I(z)$ is the total antenna current and $\zeta_0=[\mu_0/\epsilon_0]^{\frac{1}{2}}$. The effects of the antenna current and of the screen diameter may thus be evaluated separately, and some numerical values are briefly discussed. *M. C. Gray.*

King, Ronold. Asymmetrically driven antennas and the sleeve dipole. Proc. I.R.E. 38, 1154–1164 (1950).

The author applies the Hallén integral equation method to obtain approximate formulas for the currents in each arm of an asymmetrically driven antenna, assuming that the driving voltage is represented by a discontinuity in the scalar potential. For each current the usual type of series expansion is derived, in terms of the parameter already used by King and Middleton [Quart. Appl. Math. 3, 302–335; 4, 199–200 (1946); these Rev. 7, 401, 535]. The first-order terms are evaluated in terms of generalized sine and cosine integrals, but numerical computations are not carried out. Instead, the author shows that approximate evaluations can be obtained from the earlier calculations for the center driven antenna. It is shown that the input impedance is a series combination of the impedances characteristic of each arm in presence of the other, and that, by a suitable choice of the lengths of the two arms, it is possible to give the asymmetric antenna desirable broad-band characteristics.

The author next discusses applications of his formulas to the theory of the sleeve dipole, which consists essentially of a coaxial line above a horizontal conducting plane, with the inner conductor extended vertically beyond the outer. Application of the theory of images yields a symmetric structure in free space with an antenna driven internally at two points equidistant from the center. In turn, this may be represented by the sum of two asymmetric antennas, each fed at only one of the feed-points. Thus approximate formulas for the current and impedance of the sleeve dipole can be obtained by adding solutions for two asymmetric antennas. The impedance so obtained should be modified to take into account the actual conditions at the driving point (the termination of the outer conductor), where physical considerations suggest the addition of a lumped capacitance in parallel with the impedance. Curves of the variation of the sleeve dipole impedance with frequency show that this impedance is much less sensitive to frequency variation than that of an ordinary half-wave or full-wave dipole.

M. C. Gray (Murray Hill, N. J.).

King, Ronold. Theory of V-antennas. J. Appl. Phys. 22, 1111–1121 (1951).

The author considers a V-antenna consisting of two conductors of equal length and radius, one oriented along the z-axis and the other along an s-axis in the xz-plane with positive direction defined by $s=z\cos\Delta+x\sin\Delta$, where Δ is the angle between the antenna arms. It is assumed that a scalar potential difference is maintained across the base of the antenna. The integral equation for the current in the z-arm is obtained in the usual way, and resembles that for the center-driven antenna, with a modified kernel and an additional term arising from the current in the s-arm. Formulas for the functions in the first-order approximations to the current and impedance are written out, in terms of rather complicated generalizations of the sine and cosine integrals, and a curve shows the impedance of an infinitely thin V-antenna with arm-length equal to $\lambda/4$ as a function of the interior angle Δ. The remainder of the paper discusses actual methods of feeding the antenna from a balanced two-wire line, and suggests the type of correction that must be applied to the theoretical solution for an ideal slice generator. Theoretical and experimental impedance curves for different orientations of the V-antenna with respect to the two-wire line are included. *M. C. Gray.*

Casimir, H. B. G. On the theory of electromagnetic waves in resonant cavities. Philips Research Rep. 6, 162–182 (1951).

This is an expository article on the general theory of resonance. The author first discusses the resonance properties of a simple L-C circuit, of two coupled L-C circuits and of a cyclic arrangement of identical L-C circuits, with special emphasis on the energy concepts. Then he shows that similar energy formulas hold for resonant cavities. The theory of coupled cavities is outlined and a perturbation method applied to cavities with small irregularities. Finally, the author points out that the theory of standing waves in cavities was first applied half a century ago to the problem of thermal equilibrium between matter and radiation, and suggests that the modern quantum theory of permissible energy levels leads to a possible explanation of the zero-point energy of empty space. *M. C. Gray.*

Borgnis, Fritz. Über die Bedeutung der Leitungsgleichungen und des Wellenwiderstands für beliebige Wellentypen auf zylindrischen Leitungen. Arch. Elektr. Übertragung 5, 181–189 (1951).

A *TEM* wave, or Lecher wave, in a coaxial line may be defined by a transverse voltage $V(z)$ and a longitudinal current $J(z)$ which satisfy the usual transducer equations. The "wave impedance" or characteristic impedance of the coaxial line is defined by the voltage/current ratio for zero reflection at the end of the line. The author suggests that for any given mode of propagation in a cylindrical waveguide of arbitrary cross-section similar concepts of fictitious voltage, fictitious current and wave impedance may be introduced in such a way that the propagation equations in the guide are again the transducer equations. The functions V and J are not uniquely defined but one of them may be given some convenient definition (voltage between two points in a cross-section or current obtained by integrating H round a perimeter) and then the other is determined by imposing the condition that the average power flow through the guide, obtained from the field components, should also have the value $\frac{1}{2}\operatorname{Re}(VJ^*)$. The wave impedance for the guide is then determined from the voltage/current ratio. As an illustration, formulas are derived for a wave of H_{01} type (TE_{01} wave) in a rectangular guide. *M. C. Gray.*

Kleinwächter, Hans. Die Wellenausbreitung in zylindrischen Hohlleitern und die Hertzsche Lösung als Sonderfälle der Wellenausbreitung in trichterförmigen Hohlleitern. Arch. Elektr. Übertragung 5, 231–236 (1951).

The author first derives the well-known solution for E-type spherical waves (TM waves) in an infinitely long cone with vertex at the origin of a spherical coordinate system and cross-section defined by a circle $\theta=\theta_0$ on the sphere [see, for instance, S. A. Schelkunoff, Electromagnetic waves, Van Nostrand, New York, 1943, p. 399]. Then he shows that as $\theta_0\to 0$ this solution reduces, at large distances from the

origin, to the corresponding E-wave in a circular cylinder, while for $\theta_0=90°$ it reduces to the field of a current element above a perfectly conducting plane. Similarly, if the cross-section of the cone is approximately rectangular, defined by two meridians and by two circles of parallel, then at large distances each E-wave becomes the corresponding wave in a rectangular waveguide. *M. C. Gray.*

Takahashi, I., Watanabe, T., and Tanimoto, K. On the effect of a conducting screen with concentric aperture in the circular wave guide. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **26**, 123–130 (1950).

This is a discussion of the reflection and transmission properties of a small iris, based on static-field approximations.

C. J. Bouwkamp (Eindhoven).

Stenzel, Heinrich. Die Darstellung des Strahlungsfeldes zweier Strahler durch die Kurven konstanter Phase und die Kurven konstanter Amplitude. Arch. Elektr. Übertragung **5**, 517–526 (1951).

Krishna Prasad, K. V. On the approximate solutions of Maxwell's equations in an infinite medium with regions of finite conductivity. Indian J. Phys. **25**, 403–407 (1951).

Investigation of electromagnetic waves through a circular tube for the case of semiconductors. The author writes Maxwell's equations in cylindrical coördinates and finds solutions of the normal mode type expressed in Bessel and Hankel functions. Confining, for simplification, the discussion to transverse magnetic waves having radial symmetry, he obtains from the continuity of the tangential components of the magnetic force the equation for the characteristic roots

$$k_1^2J_1(u)/\mu_1uJ_0(u)=k_2^2H_1^{(1)}(v)/\mu_2vH_0^{(1)}(v),$$

where k_1 and k_2 are the propagation constants inside and outside the tube, μ_1 and μ_2 the permeabilities, $u=\lambda_1a$, $v=\lambda_2a$, a is the radius of the tube, and $\lambda_1^2-k_1^2=\lambda_2^2-k_2^2=h^2$.

When the conductivities of both the media, inside as well as outside are zero, there are no characteristic roots if $k_1<k_2$ and only a limited number if $k_1>k_2$ in agreement with a result of Hondros and Debye [Ann. Physik (4) **32**(337), 465–476 (1910)]. When the outside medium is a metal, so that k_2 is very large, then for the lower order modes v_n has a positive imaginary part, as required, but further on the imaginary part may or may not be positive. The number of permissible normal mode solutions is small for dielectric wave guides, large for metallic wave guides but always finite. So these normal mode solutions never form a complete set and the normal methods of treatment are not adequate to describe the complete field of a given source. The author announces the exact solution will be in another paper.

H. Bremekamp (Delft.).

Zernov, N. V. On the solution of unsteady boundary problems of electrodynamics. Doklady Akad. Nauk SSSR (N.S.) **80**, 33–35 (1951). (Russian)

The problem appears to be one of a sinusoidal wave, commencing at time $t=0$, impinging upon a coordinate surface of general form. The author shows that a Fourier transform reduces this non-stationary problem to a stationary one. He concludes with some remarks on the possibility of making deductions about the non-stationary case when solutions of the stationary problem are known. *F. V. Atkinson.*

Smith, P. D. P. Artificial field equations for a region where μ and ϵ vary with position. J. Appl. Phys. **21**, 1140–1149 (1950).

The author first derives very general expressions for Maxwell's equations in tensor form, assuming that μ and ϵ are both functions of position. The field components can be expressed in terms of a four-potential φ_i in a metric $ds^2=g_{ij}dx^idx^j$. Further, in regions of zero conductivity, there is complete symmetry between the electric and magnetic fields when μ and ϵ are interchanged, so that it is possible to define a new four-potential ψ_i as the counterpart of φ_i.

For practical applications the author considers cylindrical polar coordinates (r, θ, z) and assumes that the potential φ has only one non-vanishing component, in the direction of one of the coordinate axes. There are thus three possible types of TE waves in an arbitrary medium, corresponding to an electric intensity parallel to one of the three axes. For TM waves, derived from ψ, there is only one non-vanishing component of B, and hence we can only have TM waves when $\psi\equiv\psi_\theta$. TEM waves may be obtained when the potential depends on only one coordinate. The author restricts his analysis to special cases in which φ, μ and ϵ are functions only of the transverse coordinates, and obtains conditions on these functions such that the variables in the propagation equation are separable. Then he attacks the problem of determining possible forms of μ and ϵ that will give solutions of the separated equations in terms of known functions. A variety of interesting problems is solved by this procedure; e.g. a metal horn defined by $\theta=\theta_1$, $\theta=\theta_2$, with permeability $\mu=a(r)\theta^{-n}$, where a is an arbitrary function, has a potential φ_z expressed in terms of Bessel functions of order $\frac{1}{2}(n-1)$. In a coaxial line with $\varphi=\varphi_\theta$, Bessel function solutions are obtained when $\mu=Mr^{2k-2}e^{2nz}$, $\epsilon=\epsilon_0\mu_0/\mu$, with exponential propagation in the z-direction; and this TE wave can be transformed to give a similar type of TM wave. Finally the author considers waves in a region of constant μ and varying ϵ, and in particular the case of $\varphi\equiv\varphi_\theta$, and ϵ a function of r only. When $\epsilon=\epsilon_0(r_0/r)^2$ the potential in a coaxial line is

$$\varphi=[AJ_{iN}(pr)+BN_{iN}(pr)]e^{\pm pz}, \quad N=\omega r_0(\mu\epsilon_0)^{\frac{1}{2}}/c.$$

The relation between $\Gamma=ip$ and N is discussed in detail.

M. C. Gray (Murray Hill, N. J.).

Hristov, Hr. Ya. On the passage of electromagnetic waves through a plane-parallel crystal plate. Doklady Akad. Nauk SSSR (N.S.) **81**, 553–556 (1951). (Russian)

The crystal is replaced by a lattice of dipoles with electric dipole moment all in a fixed direction. The incoming wave is normal to the plate. It is assumed that all dipoles in the nth parallel layer are excited with the same amplitude Z_n. A set of N linear equations for the Z_n is obtained, where N is the number of dipole layers in the plate. Explicit formulas for the electromagnetic field quantities in terms of Z_n are given. The author asserts that his method improves on previous treatments [cf. Handbuch der Physik, 2d ed., v. 24, part 2, Springer, Berlin, 1933, pp. 770–794] by avoiding some unpleasant infinite series whose convergence is dubious.

A. J. Coleman (Toronto, Ont.).

Luttinger, J. M. The effect of a magnetic field on electrons in a periodic potential. Physical Rev. (2) **84**, 814–817 (1951).

Wannier's method [Physical Rev. (2) **52**, 191–197 (1937)] for treating the motion of electrons in a periodic electric field

is extended to cover the effect of a slowly varying magnetic field. *C. Kikuchi* (Upton, N. Y.).

Giambiagi, Juan Jose. Application of Hadamard's method to the calculation of the electromagnetic field of the electron. Revista Unión Mat. Argentina 15, 24–31 (1951). (Spanish)

The author derives the solution of the equation $\Box\phi_i=4\pi j_i$ in the form of the "logarithmic part" of an integral, following Courant-Hilbert, Methoden der mathematischen Physik, vol. II, pp. 430ff. [Springer, Berlin, 1937]. Applying the results to the case of a moving point electron, he arrives at the expressions for the potential and field of the electron previously given by Dirac [Proc. Roy. Soc. London. Ser. A. 167, 148–169 (1938)] and Fremberg [ibid. 188, 18–31 (1946); these Rev. 8, 302] by different methods.
F. John (New York, N. Y.).

Prim, R. C., III. Some results concerning the partial differential equations describing the flow of holes and electrons in semiconductors. Bell System Tech. J. 30, 1174–1213 (1951).

This paper is concerned with the system of equations describing the flow of holes and electrons in the interior of a homogeneous semiconductor, subject to the assumption of constant temperature, electrical neutrality and constant difference in concentrations of ionized donor and acceptor centers. The study is directed toward the deduction of general properties of the flow fields inside semiconductors and chiefly to the discovery of families of exact solutions of the flow equations involving arbitrary constants or preferably arbitrary functions. This is done without reference to any preconceived boundary value problems. When once a pool of such families of solutions is available, the object is to find boundary value problems of interest consistent with any of the solutions in hand ("inverse method"). After the deduction of some properties of the current density vector fields, embodied in 9 theorems, the author puts the partial differential equations restricting V, the potential of the electric intensity field, and P, the total carrier concentration, into the form

$$\operatorname{div}\operatorname{grad}(NV-(kT/e)P)=-\alpha(R-\tfrac{1}{2}\partial P/\partial t),$$
$$\operatorname{div}(P\operatorname{grad}V)=\beta(R-\tfrac{1}{2}\partial P/\partial t),$$

where R the recombination rate function is to be regarded as a function of P, $N=n+p$, n and p the concentrations of negative and positive carriers, T is the absolute temperature, e the electronic charge, α and β are constants depending on the mobility of the carriers. These are the equations to which the subsequent considerations chiefly refer. In cases where $N\neq 0$, instead of V and P the author frequently introduces new dependant variables U and K, defined by $U=(kT/eN)P$, $K=V-U$.

In the next section the author gives a summary (which he calls skimpy, but which might prove rather useful) of the results obtained in the rest of the paper. It contains however only the forms to which the solutions reduce when recombination and time variation are excluded. For the case $N=0$ the first of the above equations contains only P. The author derives an analogous but far more complicated equation containing only K for the case $N\neq 0$, $\partial U/\partial t=0$. In the last two sections there are given sample applications of some of the previous results, especially the solution for the spherically symmetric flow field without recombination. In the absence of surface recombination this provides also the hemispherically symmetric flow field of a point contact on a plane surface and remote from other electrodes or surfaces. The solution is based on a harmonic function K, for which, in view of the spherical symmetry, $K=L/r+M$ is chosen. More complicated problems may be attacked by other choices. *H. Bremekamp* (Delft).

✓ ★**Melvin, M. Avramy. Symmetry and affinity of electromagnetic fields, charges, and poles.** Proc. Second Canadian Math. Congress, Vancouver, 1949, pp. 225–255. University of Toronto Press, Toronto, 1951. $6.00.

Rikitake, Tsuneji. A method of studying the distribution of electric currents in a spherical shell having non-uniform conductivity. Bull. Earthquake Res. Inst. Tokyo 26, 11–15 (1 plate) (1948).

An approximate method of studying the distribution of electric currents in a thin spherical shell having non uniform conductivity is described. The current function is expressed by a sum of spherical surface harmonics. Numerical tables containing the values of a number of the occurring expansion coefficients are given. *F. Oberhettinger.*

Landsberg, Max. Zur Theorie und Berechnung des elektrostatischen Durchgriffs der ebenen und zylindrischen Dreipolröhre im Falle zweidimensionaler Potentialverhältnisse. I. Z. Angew. Math. Physik 2, 375–393 (1951).

In order to approximate the potential distribution in a plane triode with non-negligible radii of the grid wires, the author selects an elementary strip and develops the potential function into a series of Weierstrass zeta-functions. He then defines the potential on the grid wire surface to be constant except for a small high-order ripple composed of the terms of the Fourier series starting with order number $n+1$. The resulting system of algebraic equations for the lower Fourier coefficients can then be solved. From the potential solution, an expression for the amplification factor (or its inverse) is obtained. It is demonstrated that for very small grid wire radius the usual simple approximation formulas can be deduced but that the more exact treatment leads to significant deviations even in first approximation. The same approach is followed in the case of the cylindrical triode by first transforming, by means of the logarithmic function, the cylindrical geometry into the equivalent plane structure. The author also indicates the solution of the infinite system of equations which leads to the exact potential function. *E. Weber* (Brooklyn, N. Y.).

Neugebauer, Th. Über einen Zusammenhang zwischen Gravitation und Magnetismus. Acta Phys. Acad. Sci. Hungaricae 1, 151–165 (1951). (German. Russian summary)

It is shown that in closely packed matter (solids or liquids) the differential effect of the gravitational field on the core and on the outer electrons of an atom will induce a small polarization electric field. This polarization field will have the consequence of slightly displacing the core with respect to the outer electrons; and if the body is rotating, a net magnetic moment will result. On the assumption that in each spherical shell there is equality between the electric and the gravitational fields ($Mg=neE$; M is the mass of the atom, g the value of gravity, n the number of the outer electrons, e the electronic charge and E the electric field) and between the electric and gravitational potential energies ($\frac{1}{2}\alpha E^2=\frac{1}{2}Mgl$; α is the polarizability of the core and l is the displacement of the outer electrons relative to this core),

the author shows that the resulting magnetic moment is given by

$$\mathfrak{M}=\frac{\pi^2 N M n a g_0}{60 c e T} R_0^4$$

where g_0, R_0 and T denote the surface gravity, radius and period of rotation of the body (respectively), c is the velocity of light and N is the number of atoms per unit volume. For the case of the earth, the foregoing expression gives $\mathfrak{M}\sim 10^9$ e.s.u.; this has to be contrasted with the observed magnetic moment of 8×10^{25} e.s.u. However, the author considers it possible that a small magnetic moment of this amount may get amplified by a mechanism similar to that one encounters in ferro-electricity. *S. Chandrasekhar.*

Quantum Mechanics

Van Hove, Léon. Sur le problème des relations entre les transformations unitaires de la mécanique quantique et les transformations canoniques de la mécanique classique. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **37**, 610–620 (1951).

The author considers the mathematical formulation of the correspondence between observables in classical and quantum mechanics. His approach is essentially to define a basic group for each type of mechanics and to investigate possible isomorphisms between these groups or their subgroups. He considers systems with a finite number of degrees of freedom and for quantum mechanics takes as the basic group U the unitary operators on a Hilbert space, as usual, while for classical mechanics he takes not the usual group of all contact transformations but a larger group G with a one-dimensional center, whose quotient modulo this center is essentially the group of contact transformations. This larger group is the set of all transformations on the variables $(s, p_1, \cdots, p_n, q_1, \cdots, q_n)$ leaving invariant the form $ds-\sum p_j dq_j$; here the p_j and q_j are n pairs of canonically conjugate variables and s is an action variable. The use of this group is justified mainly by the circumstance that there is a natural Lie isomorphism between the Lie algebra of G and the set of all functions of the p_j and q_j, the latter constituting a Lie algebra under the Poisson bracket as multiplication.

While this is helpful in setting up a parallelism between the general features of the two theories, a definite correspondence between them depends on establishing an isomorphism between the two groups whose differential carries each classical p_j or q_j into the corresponding quantum-mechanical variable P_j or Q_j. The existence of any such isomorphism is very doubtful. On the other hand, if L is the normalizer of the subgroup of G whose Lie algebra arises from the linear non-homogeneous forms in the p_j and q_j, then every element of the Lie algebra of L arises from a function f of the form

$$f(p,q)=\sum a_{jk}p_jp_k+\sum b_{jk}q_jq_k+\sum c_{jk}p_jq_k+\sum a_jp_j+\sum b_jq_j+d,$$

where $a_{jk}=a_{kj}$ and $b_{jk}=b_{kj}$, and if this element is made to correspond to the operator

$$\sum a_{jk}P_jP_k+\sum b_{jk}Q_jQ_k \\ +\sum c_{jk}(P_jQ_k+Q_kP_j)+\sum a_jP_j+\sum b_jQ_j+d,$$

the result is an isomorphism of the Lie algebra of L into the Lie algebra of U. This formalizes the well known analogy between the two theories for the case when the Hamiltonian is quadratic in the p and q. The use of L affords a definite though limited scheme of quantization once a specific set of p_j and q_j are selected, but although this is probably the most rational method of quantization available, it is not invariant under arbitrary changes in the coordinate system.

I. E. Segal (Chicago, Ill.).

★**Dirac, P. A. M. The relation of classical to quantum mechanics.** Proc. Second Canadian Math. Congress, Vancouver, 1949, pp. 10–31. University of Toronto Press, Toronto, 1951. $6.00.

This describes, with a minimum of formulas, the unsatisfactory features of the present state of quantum mechanics, and discusses possible ways of avoiding the difficulties. The difficulties are due to (1) infinities which arise from treating charges as points; these are partially overcome in recent advances in method due to Lamb, Schwinger, Bethe and Feynman; (2) non-commutativity of multiplication of observables, which makes it hard to apply techniques of analysis such as differentiation and contact transformation; (3) the necessity of reconciling quantum mechanics with relativity. The author gives no final solution of these problems. He suggests that new viewpoints in classical mechanics are needed before an attempt is made to get a quantum mechanics which is relativistically invariant.

Along these lines he describes his recent efforts to replace the non-relativistic notion of an instant of time (represented by a flat three-space in four-dimensional space-time) by other forms. These include the general surface form, with a general curved three-space taking the place of an instant; the point form, where hyperbolic surfaces or hypersurfaces determined by the light-cone of a point correspond to the instants; and the front form, where this role is played by the moving front of a plane light wave. The author conjectures that some new universal physical constant with the dimensions of length will be an important feature of the next successful elaboration of quantum mechanics.

O. Frink (State College, Pa.).

★**March, Arthur. Quantum Mechanics of Particles and Wave Fields.** John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1951. x+292 pp. $5.50.

Of this book five chapters discuss non-relativistic quantum-mechanics, four deal with relativistic wave-equations, quantum electrodynamics, nuclear forces, etc., and the last expounds the author's views on removal of divergencies from self-energies etc. by the introduction of a minimum length. Most chapters end with problems for the reader. In the introduction to wave-mechanics the approach is formal and devotes little attention to basic experimental results. The book makes a brave attempt to cover a wide and, in many aspects, a difficult field. It is inevitable that sometimes brevity may lead to obscurity. Among points noted by the reviewer are the following omissions: some mention of boundary conditions in the treatment of spherical harmonics (p. 63); justification for rejection of an infinite series in the harmonic oscillator eigenfunction (p. 66); and misstatements: "for very accurately measured momentum the wave packet contracts nearly to a point . . ." (p. 44); "there are i places in which the electron is never found" (i nodes of harmonic oscillator wave-function, p. 61); "radioactive decay can be understood only on the assumption that nuclear particles also emit charged mesons" (p. 245). The representation $(-h/i)$grad for momentum is used. On pp. 231, 288 there is no indication of the distinctive properties of π and μ mesons and the consequent removal of

some difficulties. There is much that is attractive about the book and the reviewer would hope that in a second edition, by some modifications, the book would increase its value to those who need both the more elementary initial parts of the text and the last more difficult parts. *C. Strachan.*

Friedrichs, K. O. Mathematical aspects of the quantum theory of fields. I, II. Comm. Pure Appl. Math. **4**, 161–224 (1951).

These are the first two parts of a treatise on quantum field theory; four more parts are to follow. The author addresses himself "to mathematicians who are familiar with the fundamental concepts of quantum theory of single particles and who would like to learn which mathematical concepts are involved in the simplest problems of field quantum theory." His aim is merely to explain in mathematically precise terms what the physicists have been doing. But in order to explain precisely, he has also been compelled to re-formulate the theory rather completely from the beginning, and to introduce a variety of new notations and concepts.

In the first two parts he develops only the theory of a free wave-field without interaction, quantized according to Bose statistics. The chief departure from the usual formulation of the theory is that he avoids using field-operators defined at points, and instead uses operators which are formally the integrals of point field-operators, multiplied by suitable functions of position and integrated over space-time. In order to prove the mathematical consistency of these operations, he finally reduces everything to a particle representation, in which the state of the field is specified by ordinary wave-functions describing a finite number of particles; this shows that the field theory is equivalent to the theory of many indistinguishable particles, obtained from a one-particle theory by the old-fashioned method of "second quantization". He also sets up an alternative representation of the field in terms of "Hermite polynomial functionals", a generalization of ordinary Hermite polynomials to infinitely many variables. *F. J. Dyson.*

Rodičev, V. I. Some results of the general theory of fields. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **21**, 869–878 (1951). (Russian)

Vector fields for mesons of fixed rest-mass are generalized to fields which represent a superposition of fields of continuously varying rest-mass. This is accomplished by having the field depend on five field variables A_r ($r=0$–4) which are functions of five variables x^r. The variables A_r and x^r for $r=1, 2, 3, 4$ are four-vectors while A_0 and x^0 are pseudoscalars with respect to Lorentz transformations. x is the variable conjugate to the rest-mass. Field variables satisfy a five-dimensional wave equation. Single-valuedness of its solutions implies that the field can exist only in states which are quantized with respect to the rest-mass distribution. Formulas for the components of the energy-momentum tensor and related quantities are obtained.

A. J. Coleman (Toronto, Ont.).

Schwinger, Julian. The theory of quantized fields. I. Physical Rev. (2) **82**, 914–927 (1951).

A quantum dynamical principle is stated which implies equations of motion and commutation relations. For a localizable field the temporal development of a system is described by a transformation function related to eigenvectors associated with successive space-like surfaces σ_1 and σ_2. The variation of the operator U_{12} describing this development satisfies a relation $\delta U_{12}^{-1}=(i/\hbar)U_{12}^{-1}\delta W_{12}$ where $\delta W_{13}=\delta W_{12}+\delta W_{23}$. It is assumed that W_{12} is an action integral operator. Invariance of the action integral leads to conservation equations. For a given dynamical system $\delta W_{12}=F(\sigma_1)-F(\sigma_2)$, where the F are infinitesimal generating operators constructed from variables belonging to σ_1, σ_2. This gives the equations of motion. The generating operator on σ together with a principle of invariance for time-reflexion gives commutation or anti-commutation relations. Time-reflexion is included in coordinate transformations by means of a non-unitary transformation replacing a state-vector by its dual. The fundamental dynamical principle then gives the connexion between spin and statistics. *C. Strachan.*

Schwinger, Julian. On the Green's functions of quantized fields. I, II. Proc. Nat. Acad. Sci. U. S. A. **37**, 452–455, 455–459 (1951).

A preliminary and somewhat condensed account is given of a general theory of Green's functions as used in the temporal development of quantized fields. A quantum dynamical principle developed by the author in the paper reviewed above is used.

C. Strachan (Aberdeen).

Heisenberg, W. Zur Frage der Kausalität in der Quantentheorie der Elementarteilchen. Z. Naturforschung **6a**, 281–284 (1951).

In the author's theory of elementary particles [same Z. **5a**, 251–259, 367–373 (1950); these Rev. **12**, 573] there appear two types of particles which are given the names α and β. The β particles are able in certain circumstances to behave in an uncausal manner, appearing to be created at one point of space-time and absorbed at another earlier point. It was the author's original contention that the uncausal behaviour would only extend over time intervals of the order 10^{-23} seconds, and hence be physically unobjectionable. But Fierz [Helvetica Phys. Acta **23**, 731–739 (1950); these Rev. **12**, 573] has shown that on the contrary, in general the uncausal behaviour can extend over arbitrarily long times. Now the author proposes a mechanism by which the uncausal behaviour can be effectively confined to intervals of the order 10^{-23} seconds. It is only necessary to suppose, that every β particle has more than 3 times the mass of the lightest type of α particle, and will spontaneously decay into three α particles with a life-time of the order 10^{-23} sec. The author shows that the decay into three α particles will always occur before the creation of the β particle, but during a time interval so short that the uncausal sequence of events will be hardly observable. In a final remark, he observes that some degree of uncausal behaviour will be necessary in any theory in which a particle of zero rest-mass (e.g. a photon) is supposed to be a composite structure built up from particles of spin $\frac{1}{2}$ having a mutual interaction.

F. J. Dyson (Ithaca, N. Y.).